Chapter 11 in Miller and Childers

DEF A linear time invariant (LTI) system with insignal $x(t)$ and outsignal $y(t) = (Tx)(t)$ satisfies

* $(T(\alpha x_1 + \beta x_2))(t) = \alpha (Tx_1)(t) + \beta (Tx_2)(t)$
* $(T(x(\cdot - t_0)))(t) = (Tx)(t - t_0)$

DEF The impulse response of an LTI system is $h(t) = (T\delta)(t)$.

THM For an LTI system we have

$$(Tx)(t) = (h * x)(t) = \begin{cases} \int_{-\infty}^{+\infty} h(t-u)x(u)\,du & \text{continuous time} \\ \sum_{k=-\infty}^{+\infty} h(t-k)\,x(k) & \text{discrete time} \end{cases}$$

Proof (Discrete time) As $x(t) = \sum_{k=-\infty}^{+\infty} x(k)\,\delta(t-k)$ we have

$$(Tx)(t) = T\left(\sum_{k=-\infty}^{+\infty} x(k)\delta(t-k)\right) = \sum_{k=-\infty}^{+\infty} x(k)\,T(\delta(t-k)) = (h*x)(t) \quad \#$$

We will now use a WSS process $X(t)$ with mean μ_X and ACF $R_X(\tau)$ as insignal to an LTI system. For the outsignal $Y(t)$ we then obtain

$$\mu_Y(t) = E(Y(t)) = E\left(\int_{-\infty}^{+\infty} X(u)\,h(t-u)\right) = \int_{-\infty}^{+\infty} E(X(u))\,h(t-u)\,du = \mu_X \int_{-\infty}^{+\infty} h(u)\,du$$

$$R_{YY}(t,t+\tau) = E(Y(t),Y(t+\tau)) = E\left(\left(\int_{-\infty}^{+\infty} h(u)X(t-u)\,du\right)\left(\int_{-\infty}^{+\infty} h(v)\,X(t+\tau-v)\,dv\right)\right)$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(u)\,h(v)\,R_{XX}(\tau - v + u)\,du\,dv = (h(-\cdot) * h * R_{XX})(\tau) = R_{YY}(\tau)$$

$$R_{XY}(t,t+\tau) = E(X(t)Y(t+\tau)) = E\left(X(t)\int_{-\infty}^{+\infty} h(u)X(t+\tau-u)du\right)$$

$$= \int_{-\infty}^{+\infty} h(u)R_{XX}(\tau-u)\,du = (h * R_{XX})(\tau)$$

It follows that $S_{YY}(f) = |H(f)|^2 S_{XX}(f)$, $S_{XY}(f) = H(f)\,S_{XX}(f)$ and $S_{YX}(f) = \overline{H(f)}\,S_{XX}(f)$

Here we made use of the very important fact that $(\mathcal{F}(f*g)) = (\mathcal{F}f)(\mathcal{F}f)$ and $\mathcal{F}(f(-\cdot)) = \overline{\mathcal{F}f}$

Ex An AR(1)-process $Y(k) = aY(k-1) + b\,e(k)$ where $(e_k)_{k=-\infty}^{+\infty}$ is white noise can be considered as an LTI system with insignal $(e_k)_{k=-\infty}^{+\infty}$ and outsignal $Y(k)$ where

$$H(f)e(f) = (\mathcal{F}(h*e))(f) = Y(f) = a\sum_{k=-\infty}^{+\infty} e^{-j2\pi fk}\,Y(k-1) + b\,e(f)$$
$$= ae^{-j2\pi f}\,Y(f) + b\,e(f) = ae^{-j2\pi f}\,H(f)\,e(f) + b\,e(f)$$

giving $H(f) = b/(1-ae^{-j2\pi f})$ so that $S_{YY}(f) = |H(f)|^2 S_{ee}(f) = b^2\sigma^2(1+a^2-2a\cos(2\pi f))$.

The Matched Filter

A deterministic signal $s(t)$ is sent on a noisy channel.

The noise $N(t)$ is white noise.

The recived signal is $X(t) = s(t) + N(t)$.

$X(t)$ is processed through an LTI system in order to maximize the SNR $E(|(h*s)(t)|^2)/E(|(h*N)(t)|^2)$ at $t = t_0$.
Here we have

$$\frac{E((h*s)(t_0)^2)}{E((h*N)(t_0)^2)} = \frac{(h*s)(t_0)^2}{\int_{-\infty}^{+\infty}|H(f)|^2 df\,(N_0/2)} \underbrace{\leq}_{\text{Cauchy Schwarz}} \frac{2}{N_0}\frac{\left(\int_{-\infty}^{+\infty} h(u)\,s(t_0-u)\,du\right)^2}{\int_{-\infty}^{+\infty} h(u)^2\,du}$$

from which we conclude best $h(u)$ is $s(t_0-u)$ as that give equality in the above inequality.

The Wiener Filter

A WSS random process $X(t)$ is sent on a noisy channel where an independent zero-mean random noise process $N(t)$ is added so that the recived signal is $Y(t)=X(t)+N(t)$. We want to find an LTI system with frequency response $H(f)$ and out signal $Z(t)=(h*Y)(t)$ that minimizes the mean-square distance $E((Z(t)-X(t))^2)$. However

$$E((Z(t)-X(t))^2) = E\left((h*X(t)+h*N(t)-X(t))^2\right)$$

$$= E\big((h*X)(t)^2+(h*N)(t)^2+X(t)^2+2(h*X)(t)(h*N)(t)-2(h*X)(t)X(t) - 2(h*N)(t)X(t)\big)$$

$$= \int_{-\infty}^{+\infty}\left(|H(f)|^2 S_{XX}(f)+|H(f)|^2 S_{NN}(f)+S_{XX}(f)-2H(f)S_{XX}(f)\right)df$$

with minimum for

$$\frac{d}{dH(f)}\text{integrand} = 2H(f)\left(S_{XX}(f)+S_{NN}(f)\right)-2S_{XX}(f)=0 \text{ so that}$$

$$\boxed{H(f)=S_{XX}(f)/(S_{XX}(f)+S_{NN}(f))}$$