# Chapter 3 in Miller and Childers

<u>Definition</u> The <u>Cummulative Distribution Function (CDF)</u> $F_X$ of an r.v. $X$ is given by $F_X(x) = \Pr(X \le x) = \Pr(\{\xi \in S : X(\xi) \le x\})$

<u>Properties</u> ① $F_X(-\infty) = 0$, $F_X(+\infty) = 1$

② $0 \le F_X(x) = 1$

③ $F_X(x_1) \le F_X(x_2)$ for $x_1 \le x_2$

④ $\Pr(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$ for $x_1 \le x_2$

<u>Example 1 (continued)</u> for $X$ sum of two dice we have

$$F_X(x) = \begin{cases} 0 & x < 2 \\ 1/36 & 2 \le x < 3 \\ 3/36 & 3 \le x < 4 \\ 6/36 & 4 \le x < 5 \\ 10/36 & \text{for } 5 \le x < 6 \\ 15/36 & 6 \le x < 7 \\ 21/36 & 7 \le x < 8 \\ 26/36 & 8 \le x < 9 \\ 30/36 & 9 \le x < 10 \\ 33/36 & 10 \le x < 11 \\ 35/36 & 11 \le x < 12 \\ 36/36 & 12 \le x \end{cases}$$

**Definition** The Probability Density Function (PDF)

$f_X$ is given by $f_X(x) = F_X'(x)$

for a continuous random variable $X$.

**Properties** ① $f_X(x) \geq 0$ all $x$

② $F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy$

③ $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$

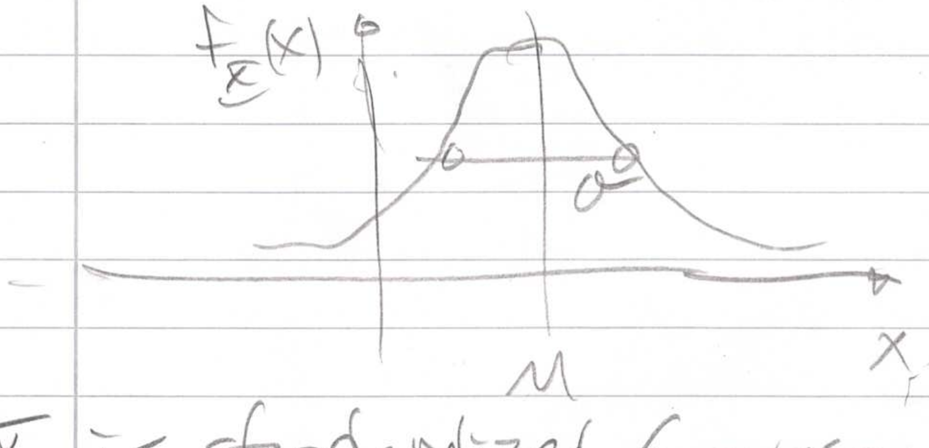
④ $\Pr(X \in A) = \int_{x \in A} f_X(x)\,dx$

$N(m, \sigma^2)$

**Gaussian r.v. $X$** is continuous with possible values $\mathbb{R}$ and

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} \text{ for } x \in \mathbb{R}$$

where $m \in \mathbb{R}$ and $\sigma^2 > 0$ are some parameters.




$X$ is standardized Gaussian $N(0,1)$

if $m = 0$ and $\sigma^2 = 1$

The CDF of a standardized Gaussian r.v. is denoted $\Phi(x)$ and the corresponding PDF $\phi(x)$ so that

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \text{ and } \Phi(x) = \int_{-\infty}^{x} \phi(y)\,dy.$$

For a general $N(m, \sigma^2)$ Gaussian r.v. $X$ we have

Theorem

$$F_X(x) = \Phi\left(\frac{x-m}{\sigma}\right)$$

Proof $F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-m)^2}{2\sigma^2}} dy$

$$= \begin{bmatrix} z = \frac{y-m}{\sigma} \\ dz = \frac{dy}{\sigma} \end{bmatrix} = \int_{-\infty}^{\frac{x-m}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \Phi\left(\frac{x-m}{\sigma}\right) \#$$

Uniform r.v. $X$ has possible values $[a,b]$ with

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

for parameters $a \le b$

so

$$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

Exponential r.v. $X$ has possible values $[0,\infty)$

with $f_X(x) = \begin{cases} \frac{1}{b} e^{-x/b} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$

for parameter $b > 0$

so that $F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-x/b} & \text{for } x \geq 0 \end{cases}$

There are many more continuous random variables but none as important as the three above.

Definition The conditional CDF of $X$ given that an event $A$ occures/occurred is $F_{X|A}(x) = \Pr(X \leq x | A) = \dfrac{\Pr((X \leq x) \cap A)}{\Pr(A)}$

The conditional CDF has same properties 1-4 as an ordinary CDF - see earlier.

Definition The conditional PDF of $X$ given that an event $A$ occures/occurred is $f_{X|A}(x) = F'_{X|A}(x)$

$f_{X|A}(x)$ has the same properties 1-4 as $f_X(x)$.

Of special interest is $F_{X|A}(x)$ for $A = a < X \le b$ in which case we set

$$F_{X|A}(x) = \frac{\Pr((X \le x) \cap (a < X \le b))}{\Pr(a < X \le b)}$$

$$= \begin{cases} 0 & x < a \\ \dfrac{F_X(x) - F_X(a)}{F_X(b) - F_X(b)} & a \le \cancel{X} \le b \\ 1 & x > b \end{cases}$$

so that

$$f_{X|A}(x) = \begin{cases} 0 & x < a \\ \dfrac{f_X(x)}{F_X(b) - F_X(a)} & a \le x \le b \\ 0 & x > b \end{cases}$$