Chapter 4 in Miller and Childers

Definition Expected value $E(X)$ of r.v.

$$\mu_X = E(X) = \begin{cases} \int_{-\infty}^{+\infty} x f_X(x)\,dx & \text{for } X \text{ continuous} \\ \sum_k p_X(k)\, k & \text{for } X \text{ discrete} \end{cases}$$

Example 1 (continued) for X sum of dice

$$E(X) = 2\cdot\frac{1}{36} + 3\cdot\frac{2}{36} + 4\cdot\frac{3}{36} + 5\cdot\frac{4}{36} + 6\cdot\frac{5}{36} + 7\cdot\frac{6}{36} + 8\cdot\frac{5}{36} + 9\cdot\frac{4}{36} + 10\cdot\frac{3}{36} + 11\cdot\frac{2}{36} + 12\cdot\frac{1}{36}$$

$$= 7$$:)

Theorem

$$E(g(X)) = \begin{cases} \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx & X \text{ continuous} \\ \sum_k g(k) P_X(k) & X \text{ discrete} \end{cases}$$

Theorem $E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$

Definition Moment $E(X^n)$, $n = 1, 2, \ldots$

Example 2 Uniform r.v.

$$E(X) = \int_a^b \frac{x}{b-a}\,dx = \left[\frac{x^2}{2(b-a)}\right]_{x=a}^{x=b} = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

Definition

Central moments $E\left((X-\mu_X)^n\right), n=1,2,\dots$

Conditional expected value

$$E(X|A) = \begin{cases} \int_{-\infty}^{+\infty} x\, f_{X|A}(x)\,dx & X \text{ continuous} \\ \sum_k k\, P_{X|A}(k) = \sum_k \Pr(X=k|A)\, k & X \text{ discrete} \end{cases}$$

Theorem $Y = g(X)$ with g increasing $\Rightarrow$

$$f_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(X \le g^{-1}(y))$$

$$= f_X(g^{-1}(y))\, \frac{d}{dy} g^{-1}(y)$$

Example X uniform over $[0,1]$ and

$Y = -\ln(X) \Rightarrow f_Y(y) = \frac{d}{dy} Pr(-\ln(X) \le y)$

$= \frac{d}{dy} Pr(X \ge e^{-y})$

$= \frac{d}{dy} \quad 1 - Pr(X \le e^{-y})$

$= - f_X(e^{-y}) \frac{d}{dy}(e^{-y}) = e^{-y}$ = exponential r.v.

for $y > 0$

Definition Characteristic Function (CHF) of r.v. X

$\Phi_X(\omega) = E(e^{j\omega X}) \underset{\uparrow}{=} \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x)\, dx$

if X continuous

$\mathcal{F}$-transform of X

Example For X $N(m, \sigma^2)$ Gaussian

$$\Phi_X(\omega) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} e^{j\omega x}\, dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m-j\omega\sigma^2)^2}{2\sigma^2}}\, dx\, e^{mj\omega - \frac{\omega^2\sigma^2}{2}}$$

$$= e^{mj\omega - \frac{\omega^2\sigma^2}{2}}$$

$$\Phi_X^{(n)}(0) = \frac{d^n}{d\omega^n}\Phi_X(\omega)\Big|_{\omega=0}$$

$$= \frac{d^n}{d\omega^n}E\left(e^{j\omega X}\right)\Big|_{\omega=0}$$

$$= E\left(\frac{d^n}{d\omega^n}e^{j\omega X}\right)\Big|_{\omega=0}$$

$$= E\left((jX)^n e^{j\omega X}\right)\Big|_{\omega=0}$$

$$= E(X^n)\, j^n$$

so $\boxed{E(X^n) = (-j)^n \Phi_X^{(n)}(0)}$

Example For Gaussian

$$-j\,\Phi_X'(0) = (-j)(mj - \omega\sigma^2)\,\Phi_X(\omega)\Big|_{\omega=0} = m$$

$$(-j^2)\,\Phi_X''(0) = (-j)^2\left((mj - \omega\sigma^2)^2 - \sigma^2\right)\Phi_X(\omega)\Big|_{\omega=0}$$

$$= m^2 + \sigma^2$$

Definition **Probability Generating Function (PGF)**

$$H_{\underline{X}}(z) = \sum_{k=0}^{\infty} P_{\underline{X}}(k) z^k \quad \text{for discrete r.v. } \underline{X} \text{ ($\mathbb{N}$-valued)}$$

$$= E(z^{\underline{X}})$$

Is not much used ..

$$H_{\underline{X}}^{(n)}(z) = \frac{d^n}{dz^n} H_{\underline{X}}(z) = \frac{d^n}{dz^n} E\left(z^{\underline{X}}\right)$$

$$= E\left(\underline{X}(\underline{X}-1)\cdot\ldots\cdot(\underline{X}-k+1) z^k\right)$$

so $H_{\underline{X}}^{(n)}(1) = E\left(\underline{X}(\underline{X}-1)\cdot\ldots\cdot(\underline{X}-k+1)\right)$

$$P_{\underline{X}}(n) = \frac{1}{n!} \frac{d^n}{dz^n} H_{\underline{X}}(z)\Big|_{z=0}$$

by Power series expansion