Chapter 5 in Miller and Childers

Def The <u>joint CDF</u> of a pair of r.v. (X, Y) is
$F_{X,Y}(x,y) = \Pr(X \le x, Y \le y)$.

<u>Properties</u> $F_{X,Y}(-\infty,-\infty) = F_{X,Y}(x,-\infty) = F_{X,Y}(-\infty,y) = 0$

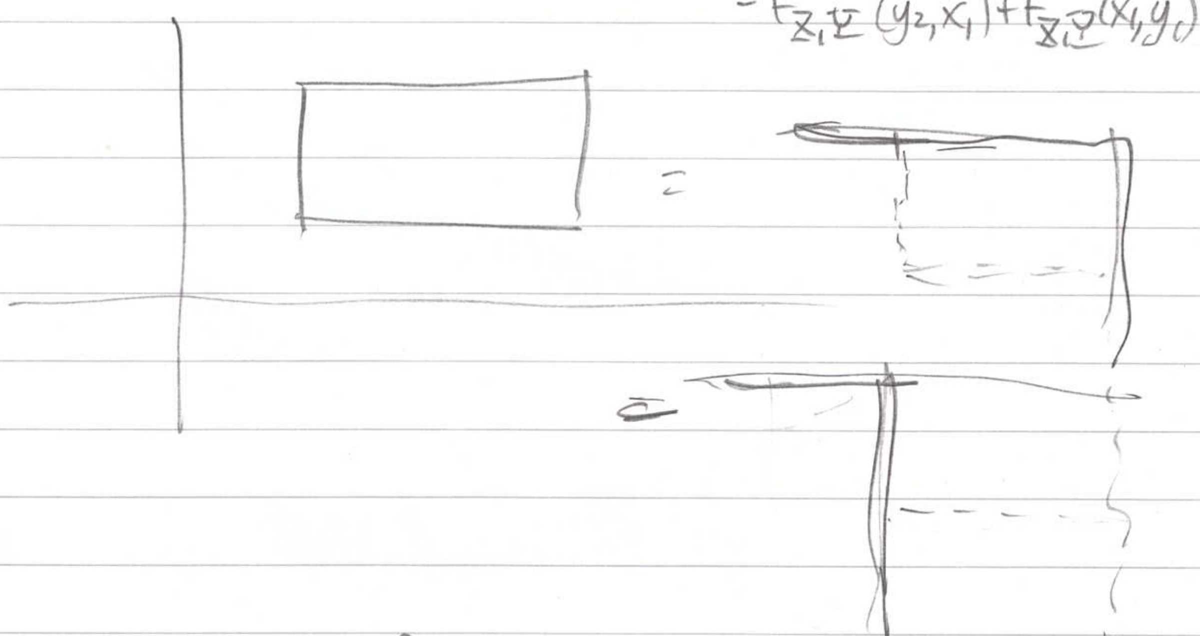
$F_{X,Y}(\infty,\infty) = 1$

$0 \le F_{X,Y}(x,y) \le 1$

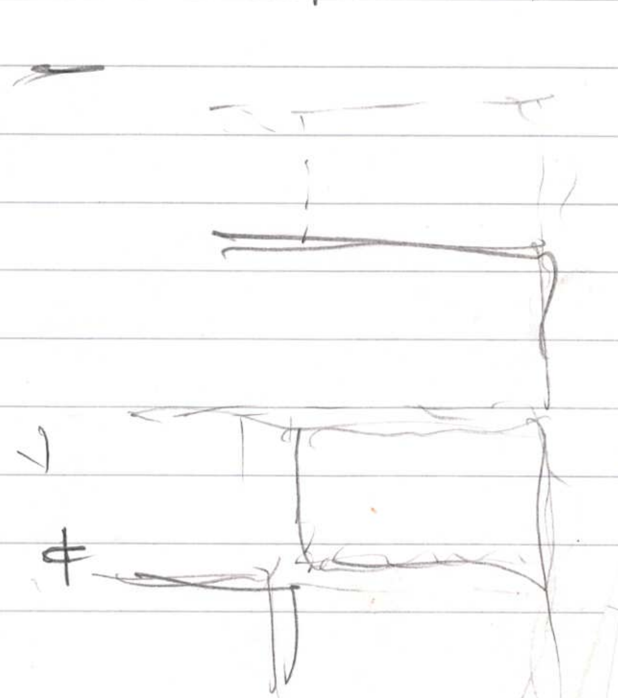
$F_{X,Y}(x,\infty) = F_X(x) \quad F_{X,Y}(\infty,y) = F_Y(y)$

$\Pr(x_1 < X \le x_2, y_1 < Y \le y_2) = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(y_2,x_1) + F_{X,Y}(x_1,y_1)$

<u>Proof</u>



Def The concepts of <u>discrete</u> and <u>continuous</u> two-dimensional r.v.'s are defined as before.



Def The PDF of a continuous r.v. (X,Y) is

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

Properties $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\,du\,dv$

$$f_{X,Y}(x,y) \geq 0$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy, \quad f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx$$

$$\Pr((X,Y)\in A) = \iint_{(x,y)\in A} f_{X,Y}(x,y)\,dx\,dy$$

Ex For $f_{X,Y}(x,y) = \frac{1}{2}e^{-x}e^{-y/2}$ for $x,y \geq 0$ we have

$$\Pr(X>Y) = \int_{y=0}^{y=+\infty}\int_{x=y}^{x=+\infty} e^{-x}e^{-y/2}\,\frac{dx\,dy}{2} = \int_{y=0}^{y=\infty} \frac{1}{2}e^{-y/2}e^{-y}\,dy = \frac{1}{3}$$

Def The PMF of a discrete r.v. (X,Y) is

$$p_{X,Y}(x,y) = \Pr(X=x, Y=y)$$

Properties $0 \leq p_{X,Y}(x,y) \leq 1$

$$\sum_{\text{all } x,y}\sum p_{X,Y}(x,y) = 1$$

$$\sum_{\text{all } x} p_{X,Y}(x,y) = p_Y(y), \quad \sum_{\text{all } y} p_{X,Y}(x,y) = p_X(x)$$

$$\Pr((X,Y)\in A) = \sum_{(x,y)\in A}\sum p_{X,Y}(x,y)$$

For (X,Y) discrete we have

$$\Pr(X=x \mid Y=y) = \frac{\Pr(X=x, Y=y)}{\Pr(Y=y)}.$$

For (X,Y) continuous we find the definition of $f_{X|Y}(x|y)$ as

$$\lim_{x,y \downarrow 0} \frac{Pr(x-dx < X \le x \mid y-dy < Y \le y)}{dx}$$

$$= \lim_{x,y \downarrow 0} \frac{Pr(x-dx < X \le x, y-dy < Y \le y)\, dy}{Pr(y-dy < Y \le y)\, dx\, dy} = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Ex For $f_{X,Y}(x,y) = \frac{1}{\pi\sqrt{3}} \exp\left(-\frac{2}{3}(x^2 - xy + y^2)\right)$ we have

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{3/4}} e^{-\frac{(y-\frac{1}{2}x)^2}{2(\sqrt{3/4})^2}} dy\, e^{-\frac{2}{3}x^2 + \frac{1}{6}x^2} \frac{\sqrt{2\pi}\sqrt{3/4}}{\pi\sqrt{3}}$$

$$= 1 \cdot e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} \quad \#$$

$$E(g(X,Y)) = \begin{cases} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy & (X,Y) \text{ continuous} \\ \sum_k \sum_l g(k,l) f_{X,Y}(k,l) & (X,Y) \text{ discrete} \end{cases}$$

Variance $\quad Var(X) = \sigma_X^2 = E((X - \overset{E(X)}{\mu_X})^2) = E(X^2) - \mu_X^2$

Correlation $\quad R_{XY} = E(XY)$

Standard deviation σ_X

Covariance $\quad Cov(X,Y) = E((X-\mu_X)(Y-\mu_Y))$

Formula $\quad Cov(X,Y) = R_{XY} - \mu_X \mu_Y$, $Cov(X,X) = Var(X)$

Proof By inspection. #

Correlation coefficient $\rho_{X,Y} = \dfrac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$

$$Cov\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \, Cov(X_i, Y_j)$$

In particular
$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \, Cov(X_i, X_j)$$

Thm $|\rho_{X,Y}| \leq 1$ Remember: $E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$

Proof $0 \leq Var\left(\dfrac{X}{\sigma_X} \pm \dfrac{Y}{\sigma_Y}\right) = 1 \pm 2\rho_{XY} + 1$ #

Def X, Y are uncorrelated if $\rho_{X,Y} = Cov(X,Y) = 0$

Joint moment $E(X^m Y^n)$

Joint central moment $E((X-\mu_X)^m (Y-\mu_Y)^n)$

$$E(g(X) \mid Y=y) = \begin{cases} \int_{-\infty}^{+\infty} g(x) f_{X|Y}(x|y)\,dx & (X,Y) \text{ continuous} \\ \sum g(k) Pr(X=k \mid Y=y) & (X,Y) \text{ discrete} \end{cases}$$

Def X and Y are independent if $Pr(X \in A, Y \in B) =$
$= Pr(X \in A) \times P(Y \in B)$

Thm X and Y are independent iff $F_{X,Y}(x,y) = F_X(x) F_Y(y)$

iff $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ // $P_{X,Y}(x,y) = P_X(x) P_Y(y)$

$(X, Y$ continuous // discrete

Thm If X, Y ar independent $Cov(X,Y) = 0$

Proof $E(XY) = \iint xy f_{X,Y}(x,y)\,dx\,dy = \iint xy f_X(x) f_Y(y)\,dy\,dx = E(X)E(Y)$ #

Def A jointly Gaussian r.v. (X, Y) has PDF

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}} \exp\left(-\frac{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho_{XY}\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}{2(1-\rho_{XY}^2)}\right)$$

Here it can be shown that $\mu_X, \mu_Y, \rho_{X,Y}, \sigma_X$ and σ_Y are means, correlation coefficient and standarddeviations.

Def Joint CHF $\Phi_{X,Y}(\omega_1, \omega_2) = E(e^{j(\omega_1 X + \omega_2 Y)})$.

Thm $E(X^m Y^n) = (-j)^{m+n} \frac{\partial^m}{\partial \omega_1^m} \frac{\partial^n}{\partial \omega_2^n} \Phi_{X,Y}(\omega_1, \omega_2) \Big|_{(\omega_1,\omega_2)=(0,0)}$

Def For (X, Y) discrete $\mathbb{N}^2$-valued joint PGF

$$H_{X,Y}(z_1, z_2) = E(z_1^X z_2^Y) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} z_1^m z_2^n p_{X,Y}(m,n)$$

Thm $p_{X,Y}(k,l) = \frac{1}{k!\,l!} \frac{\partial^{k+l}}{\partial z_1^k \partial z_2^l} H_{X,Y}(z_1,z_2) \Big|_{(z_1,z_2)=(0,0)}$

$$E\big(X(X-1)\cdot\ldots\cdot(X-k+1)Y(Y-1)\cdot\ldots\cdot(Y-l+1)\big) = \frac{\partial^{k+l}}{\partial z_1^k \partial z_1^l} H_{X,Y}(z_1,z_2)\Big|_{(z_1,z_2)=(1,1)}$$

Thm If X, Y are independent then

$$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_X(z-y) f_Y(y)\,dy = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)\,dx$$

Proof $f_{X+Y}(z) = \frac{d}{dz} P_r(X+Y \le z) = \frac{d}{dz} \iint_{x+y\le z} f_X(x) f_Y(y)\,dx\,dy$

$$= \int_{x=-\infty}^{x=+\infty} \left(\int_{y=-\infty}^{y=z-x} f_Y(y)\,dy \right) f_X(x)\,dx = \int_{-\infty}^{+\infty} f_Y(z-x) f_X(x)\,dx. \quad \#$$

Thm If $(Z, W) = (g_1(X, Y), g_2(X, Y))$ then

$$f_{Z,W}(z, w) = \frac{f_{X,Y}(x, y)}{\left| \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{pmatrix} \right|} \Bigg|_{\substack{x = h_1(z, w) \\ y = h_2(z, w)}}$$

for (X, Y) continuous

where h_1, h_2 is the inverse transform to g_1, g_2.

Proof This is just change of variable in a multidimensional integral. #