Chapter 10 in Miller and Childers

Consider a continuous time WSS process $\{X(t)\}_{t\in\mathbb{R}}$ with ACF $R_{XX}(\tau)$.

DEF The PSD is the Fourier transform of the ACF
$$S_{XX}(f)=\int_{-\infty}^{+\infty} e^{-j2\pi f\tau} R_{XX}(\tau)\,d\tau=(\mathcal{F}R_{XX})(f),\ f\in\mathbb{R}.$$

Fact By the inversion formula it follows that
$$R_{XX}(\tau)=\int_{-\infty}^{+\infty} e^{j2\pi f\tau} S_{XX}(f)\,df=(\mathcal{F}^{-1}S_{XX})(\tau).$$

Example Recall that $X(t)=A\sin(\omega_0 t+\Theta)$ with $\omega_0\in\mathbb{R}$ a constant and A and Θ independent random variables with Θ unif $[0,2\pi]$ has $R_{XX}(\tau)=\frac{1}{2}E(A^2)\cos(\omega_0\tau)$ which gives $S_{XX}(f)=\frac{1}{4}E(A^2)\left(\delta(f-\omega_0/2\pi)+\delta(f+\omega_0/2\pi)\right)$ by "the backway" method.

Example One common ACF (it turns out) is $R_{XX}(\tau)=e^{-\alpha|\tau|}$ has $S_{XX}(f)=\frac{2\alpha}{\alpha^2+4\pi^2f^2}$ since

$$\int_{-\infty}^{+\infty} e^{-j2\pi f\tau}e^{-\alpha|\tau|}\,d\tau=\int_0^{+\infty} e^{-(\alpha+j2\pi f)\tau}\,d\tau+\int_{-\infty}^{0} e^{(\alpha-j2\pi f)\tau}\,d\tau$$
$$=\left[-\frac{e^{-(\alpha+j2\pi f)\tau}}{\alpha+j2\pi f}\right]_0^{+\infty}+\left[\frac{e^{(\alpha-j2\pi f)\tau}}{\alpha-j2\pi f}\right]_{-\infty}^{0}=\frac{1}{\alpha+j2\pi f}+\frac{1}{\alpha-j2\pi f}$$

From this in turn one can conclude without calculation that $R_{XX}(\tau)=\frac{2\alpha}{\alpha^2+4\pi^2\tau^2}$ has $S_{XX}(f)=\quad e^{-\alpha|f|}$.

For a discrete time WSS process $\{X(n)\}_{n\in\mathbb{Z}}$ with ACF $R_{XX}(\tau)$ the PSD is given by

$$S_{XX}(f) = (\mathcal{F}R_{XX})(f) = \sum_{k=-\infty}^{+\infty} e^{-j2\pi f k} R_{XX}(k) \quad \text{for} \quad f \in [-1/2, 1/2].$$

By Fourier series technique it follows that

$$R_{XX}(\tau) = \int_{-1/2}^{1/2} e^{j2\pi f\tau} S_{XX}(f)\, df$$

$\boxed{\text{Properties}}$ $S_{XX}(f) \geq 0$

$S_{XX}(f) = S_{XX}(-f)$

$S_{XX}(f)$ is real

$$E(X(t)^2) = \begin{cases} \int_{-\infty}^{+\infty} S_{XX}(f)\, df & \text{continuous time} \\ \int_{-1/2}^{1/2} S_{XX}(f)\, df & \text{discrete time} \end{cases}$$

Proof: The first property is very hard to prove while the fourth is by inspection of the inversion formula. In continuous time we further have

$$S_{XX}(-f) = \int_{-\infty}^{+\infty} e^{j2\pi f\tau} R_{XX}(\tau)\, d\tau = \int_{-\infty}^{+\infty} e^{-j2\pi f\tau} R_{XX}(-\tau)\, d\tau = S_{XX}(f)$$

$$\overline{S_{XX}(f)} = S_{XX}(-f) = S_{XX}(f) \quad \#$$

$\boxed{\text{DEF}}$ The crosspectraldensity $S_{XY}(f)$ between two jointly WSS processes $X(t)$ and $Y(t)$ is defined as $S_{XY}(f) = (\mathcal{F}R_{XY})(f)$ where $R_{XY}(\tau) = E(X(t)\, Y(t+\tau))$.

The bandwith of a process is a certain measure of the width of the graph of $S_{XX}(f)$, for example, the width of the 3dB-zone. There are several ways to measure bandwidth.

Note WSS $X(t)$ and $Y(t)$ are jointly WSS if $R_{XY}(t, t+\tau)$ does not depend on t.

Given an observation $(X(t))_{t\in[-t_0,t_0]}$ of a WSS process $X(t)$ it is natural to estimate (based on ergodicity in the autocorrelation)

$$\hat{R}_{XX}(\tau) = \begin{cases} \dfrac{1}{2t_0-|\tau|} \displaystyle\int_{-t_0+|\tau|/2}^{t_0-|\tau|/2} X\left(t-\frac{|\tau|}{2}\right) X\left(t+\frac{|\tau|}{2}\right) dt & \text{for } |\tau|<2t_0 \\ 0 & \text{for } |\tau|\geq 2t_0 \end{cases}$$

In reality it turns out that it is suitable to damp the estimate of $R_{XX}(\tau)$ for big $|\tau|$ close to $2t_0$ as the are more uncertain than other because based on fewer data so one uses $\hat{R}^{(w)}_{XX}(\tau) = w(\tau)\,\hat{R}_{XX}(\tau)$

where $|w(\tau)|\leq 1$ is a so called <u>windowing function</u>.

The most commonly used windowing function is

$$w(\tau) = \mathrm{tri}\left(\frac{\tau}{2t_0}\right) = \begin{cases} 1-\dfrac{|\tau|}{2t_0} & |\tau|<2t_0 \\ 0 & |\tau|\geq 2t_0 \end{cases}$$

which in turn leads to the most commonly use ACF-estimate

$$\hat{R}^{(tri)}_{XX}(\tau) = \begin{cases} \dfrac{1}{2t_0} \displaystyle\int_{-t_0+|\tau|/2}^{t_0-|\tau|/2} X\left(t-\frac{|\tau|}{2}\right) X\left(t+\frac{|\tau|}{2}\right) d\tau & |\tau|<2t_0 \\ 0 & |\tau|\geq 2t_0 \end{cases}$$

To estimate $S_{XX}(f)$ one simply uses

$$\hat{S}^{(w)}_{XX}(f) = \left(\mathcal{F}\,\hat{R}^{(w)}_{XX}\right)(f).$$

The most commonly used PSD-estimate becomes

$$\hat{S}^{(tri)}_{XX}(f) = (\mathcal{F}\hat{R}^{(tri)}_{XX})(f).$$

We have the following useful result:

THM

$$\hat{S}^{(tri)}_{XX}(f) = \frac{1}{2t_0}|X_{t_0}(f)|^2 \text{ where}$$

$$X_{t_0}(f) = (\mathcal{F}X_{t_0})(f) = \int_{-t_0}^{t_0} e^{-j2\pi f t} X(t)\,dt.$$

DEF $\frac{1}{2t_0}|X_{t_0}(f)|^2$ is called <u>the periodogram</u> and denoted $\tilde{S}^{(per)}_{XX}(f)$.

<u>Proof</u>

$$\frac{1}{2t_0}|X_{t_0}(f)|^2 = \frac{1}{2t_0}\mathcal{F}[X_{t_0}(\tau) * X_{t_0}(-\tau)](f)$$

$$= \frac{1}{2t_0}\mathcal{F}\left[\int_{-\infty}^{+\infty} X_{t_0}(u)\, X_{t_0}(u-\tau)\,du\right](f)$$

$$= \frac{1}{2t_0}\mathcal{F}\left[\int_{-t_0}^{t_0} X_{t_0}(u)\, X_{t_0}(u-\tau)\,du\right](f)$$

$$= \frac{1}{2t_0}\mathcal{F}\left[\int_{-t_0+|\tau|/2}^{t_0-|\tau|/2} X(t-|\tau|/2)\, X(t+|\tau|/2)\,dt\right](f)$$

$$= (\mathcal{F}\hat{R}^{(tri)}_{XX})(f) = \hat{S}^{(tri)}_{XX}(f) \quad \#$$

The above mentioned method to estimate PSD is called non-parametric because it does not assume any advance knowledge about the process at hand whatsoever.

Now let $\{e[n]\}_{n=-\infty}^{+\infty}$ be IID zero-mean with variance σ^2 (= discret white noise) and consider a process $\{Y[n]\}_{n=-\infty}^{+\infty}$ given by

$$Y[n] = \sum_{i=1}^{p} a_i Y[n-i] + \sum_{i=0}^{q} b_i e[n-i]$$

where $e[n]$ independent of ~~earlier~~ earlier $Y[n]$

This process is called an ARMA(p,q)-process. When p=0 it is called an MA(q)-process and when q=1 it is called an AR(p)-process.

Let us study the AR(1)-process

$$Y[n] = a_1 Y[n-1] + e_n$$

We will see in Ch 11 that Y is WSS so

$$R_{YY}(0) = E(Y[n]^2) = E((a_1 Y[n-1] + e_n)^2) = a_1^2 R_{YY}(0) + \sigma^2$$

so that $R_{YY}(0) = \frac{\sigma^2}{1-a_1^2}$. Further

$$R_{YY}(k) = E(Y(n)Y(n+k)) \quad \text{for } k \geq 1$$

$$= E(Y[n](a_1 Y(n+k-1) + e_{n+k-1})) = a_1 R_{YY}(k-1)$$

which is a difference equation with solution

$$R_{YY}(k) = a_1^{|k|} R_{YY}(0) = a_1^{|k|} \frac{\sigma^2}{1-a_1^2}.$$

Now we can do estimates $\hat{a}_1$ and $\hat{\sigma}^2$ of a_1 and σ^2 from $\hat{R}_{YY}(0)$ and $\hat{R}_{YY}(1)$ through

$$\hat{R}_{YY}(0) = \frac{\hat{\sigma}^2}{1-\hat{a}_1^2} \text{ and } \hat{R}_{YY}(1) = \hat{a}_1 \frac{\hat{\sigma}^2}{1-\hat{a}_1^2}$$

and then do a parametric estimate of

$$S_{XX}(f) = \frac{\sigma^2}{|1-a_1 e^{-j2\pi f}|^2}$$

see Ch 11 for proof of this $S_{XX}(f)$ formula

with

$$\hat{S}_{XX}(f) = \frac{\hat{\sigma}^2}{|1-\hat{a}_1 e^{j2\pi f}|^2}$$

$\boxed{\text{DEF}}$ White noise is a WSS (possibly Gaussian) process $X(t)$ with constant PSD $S_{XX}(f) = N_0/2$.

Going the "back way" we realize that $X(t)$ must have ACF $R_{XX}(\tau) = N_0/2\ \delta(\tau)$ in both discrete and continuous time.