# Chapter 8 in Miller and Childers

**Def** A random process (also called stochastic process) is a family $X(t) = X(t, \xi)$, $\xi \in S$, of random variables indexed by time $t \in T$.

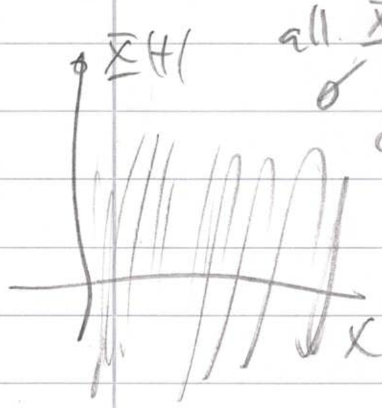
The time parameter set $T$ is either discrete $T = \mathbb{N}, \mathbb{Z}, \{0, \ldots, n\}$ etc. or continuous $T = \mathbb{R}, \mathbb{R}^+, [a, b]$ etc.

As the CDF $F_X(x)$ is used to study r.v.'s $X$ one might believe the CDF $F_{X(t)}(x)$ of a random process $X(t)$ at each time $t$ says a lot about the process.

This is not true:

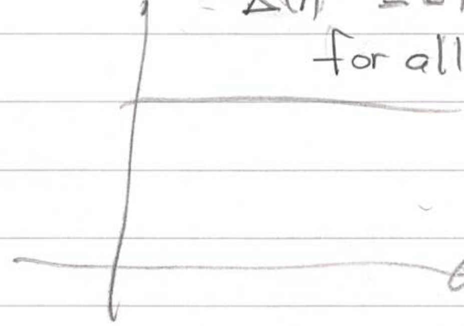
Ex 1

all $X(t)$ independent $N(0,1)$-distributed




Ex 2

$X(t) = Y \in N(0,1)$ for all $t$



Both the above processes has CDF $F_{X(t)}(x) = \Phi(x)$ for each $t \in T$ but are totally different in behaviour.

**Def** The mean function $\mu_X(t) = E(X(t))$

The autocorrelation function $R_{XX}(t_1,t_2) = E(X(t_1)X(t_2))$

The autocovariance function $C_{XX}(t_1,t_2) = \mathrm{Cov}(X(t_1), X(t_2))$

Crosscorrelation function $R_{X,Y}(t_1,t_2) = E(X(t_1)Y(t_2))$

Crosscovariance function $C_{X,Y}(t_1,t_2) = \mathrm{Cov}(X(t_1), Y(t_2))$

Example For $X(t) = U\cos(\omega t) + V\sin(\omega t)$ where $U$ and $V$ are zero mean uncorrelated with variance $\sigma^2$ we have $\mu_X(t) = 0$ and

$$R_{X,X}(s,t) = C_{X,X}(s,t) = E(U^2)\cos(\omega s)\cos(\omega t) + E(V^2)\sin(\omega s)\sin(\omega t) = \sigma^2 \cos(\omega(s-t)) \quad \#$$

$\boxed{\text{Cosine-process}}$

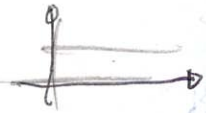
Example For $X(t) = a(\sin \omega t + \Theta)$ with $\Theta$ uniformly distributed over $[0, 2\pi]$ we have $\mu_X(t) = 0$ and

$$R_{X,X}(s,t) = C_{X,X}(s,t) = a^2 E(\sin(\omega s + \Theta)\sin(\omega t + \Theta))$$

$$= a^2 E\left(\tfrac{1}{2}\cos(\omega s - \omega t) + \tfrac{1}{2}\cos(\omega s + \omega t + 2\Theta)\right)$$

$$= \tfrac{1}{2} a^2 \cos(\omega s - \omega t) \quad \#$$

**Def** $X(t)$ is (strictly) stationary if $F_{X(t_1),\dots,X(t_n)}(x_1,\dots,x_n) = F_{X(t_1+h),\dots,X(t_n+h)}(x_1,\dots,x_n)$

Example The processes  and  are both stationary.

Def $X(t)$ is wide sense stationary (weak) WSS if

$\begin{cases} \mu_X(t) = \mu_X \\ R_{XX}(t, t+\tau) \end{cases}$ does not depend on $t$.

Thm stationary $\Rightarrow$ WSS

Notation For $X(t)$ WSS we write $R_{XX}(t, t+\tau) = R_{XX}(\tau)$.

Example Both trigonometric processes we saw earlier are WSS as are ⧸⧸⧸ and —.

Def A WSS $X(t)$ is called ergodic in the mean if

$$\lim_{t_0 \to \infty} \frac{1}{2t_0} \int_{-t_0}^{t_0} X(t)\, dt = \mu_X$$

and ergodic in the autocorrelation if

$$\lim_{t_0 \to \infty} \frac{1}{2t_0} \int_{-t_0}^{t_0} X(t) X(t+\tau)\, dt = R_{XX}(\tau)$$

EX $X(t) = a \sin(\omega t + \Theta)$ with $\Theta$ uniform over $[0, 2\pi]$ is ergodic in the mean and in the autocorrelation as

$$a^2 \sin(\omega t + \Theta) \sin(\omega(t+\tau) + \Theta)$$
$$= \frac{a^2}{2} \cos(\omega \tau) - \frac{a^2}{2} \cos(2\omega t + \omega\tau + 2\Theta)$$

Note that $\mathrm{Var}\left(\frac{1}{2t_0} \int_{-t_0}^{t_0} X(t)\, dt\right) = \frac{1}{4t_0^2} \int_{-t_0}^{t_0} \int_{-t_0}^{t_0} C_{XX}(s,t)\, ds\, dt$

so that $X(t)$ is ergodic in the mean if $\lim_{\tau \to \infty} R_{XX}(\tau) = \mu_X^2$

Properties of autocorrelation for WSS $X(t)$

* $|R_{XX}(\tau)| \leq R_{XX}(0)$ and $\left(E(X(t)X(t+\tau))\right)^2 \leq \left(E(X(t)^2)\right)\left(E(X(t+\tau)^2)\right)$

since $E\left(\left(\frac{X(t)}{\sqrt{R_{XX}(0)}} \pm \frac{X(t+\tau)}{\sqrt{R_{XX}(0)}}\right)^2\right) \geq 0$

* $R_{XX}(\tau) = R_{XX}(-\tau)$ * $R_{XX}(0) = E(X(t)^2)$

Def A random process $X(t)$ is Gaussian if $\sum_{i=1}^{n} a_i X(t_i)$ is normal for all $a_i$ and $t_i$

Note $X(t)$ Gaussian $\Rightarrow$ each $X(t)$-value normal but it is not the other way around!

Ex $X(t) = U\cos(\omega t) + V\sin(\omega t)$ with $U, V \in N(0, \sigma^2)$ independent is Gaussian.

Fact A Gaussian process is (probabilistically) determined entirely by its mean function and autocorrelation function.

Proof* $\Phi_{X(t_1),\dots,X(t_n)}(\omega_1,\dots,\omega_n) = E\left(e^{j\overbrace{\sum_i \omega_i X(t_i)}^{N(\mu,\sigma^2)}}\right) = e^{j\mu - \frac{1}{2}\sigma^2}$

where $\mu = E\left(\sum \omega_i X(t_i)\right) = \sum \omega_i \mu_X(t_i)$ and

$\sigma^2 = Var\left(\sum \omega_i X(t_i)\right) = \sum_i \sum_j \omega_i \omega_j C_{XX}(t_i, t_j)$. #

determined by $R_{XX}$ and $\mu_X$

Fact A Gaussian process is stationary if WSS.

Proof If WSS it has same $\mu_X$ and $R_{XX}$ as stationary and therefor is stationary as determined by those. #

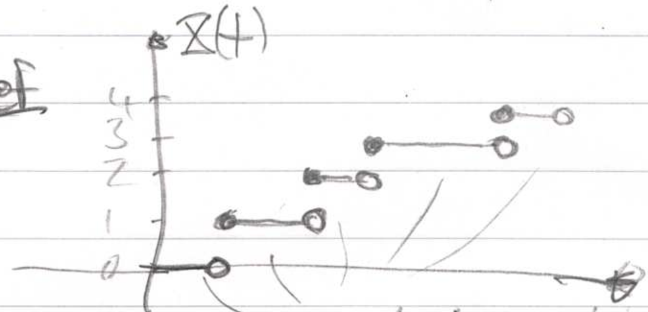
<u>Fact</u> Two Gaussian process values $X(s)$ and $X(t)$ are independent if uncorrelated.

<u>Proof</u> If uncorrelated they have same $\mu_X$ and $R_{XX}(s,t)$ as if independent and thus are independent since determined by those. #

<u>Def</u> Poisson process $X(t), t \geq 0$

1. $X(0)=0$
2. $X(t+s)-X(s) \sim P_o(\lambda t)$ for $s,t \geq 0$
3. $X(t+s)-X(s)$ independent of $(X(r))_{r\in[0,s]}$

<u>Alt def</u>




independent exp-distributed with mean $\frac{1}{\lambda}$

<u>Example</u> $\mu_X(t)=\lambda t$

$$R_{XX}(t,t+\tau)=E(X(t)X(t))+E(X(t)(X(t+\tau)-X(t)))$$
$$=E(P_o(\lambda t)^2)+E(P_o(\lambda t))E(P_o(\lambda\tau))$$
$$=\lambda t+(\lambda t)^2+\lambda t\lambda\tau$$

$P=P_r$

<u>Example</u> $$P(X(1)=1 \mid X(2)=2)=\frac{P(X(1)=1, X(2)=2)}{P(X(2)=2)}$$

$$=\frac{P(X(1)=1)\,P(X(2)-X(1)=1)}{P(X(2)=2)}=\frac{\frac{\lambda^1}{1!}e^{-\lambda}\frac{\lambda^1}{1!}e^{-\lambda}}{\frac{(2\lambda)^2}{2!}e^{-2\lambda}}=\frac{1}{2}$$

**Example** Find $\Pr(X(1)+2X(2)\geq 3)$ for cosine-process. $\boxed{8.6}$
with $U$ and $V$ Gaussian/normal.

**Solution** $X(1)+2X(2)$ is $N(0,\Sigma^2)$ with

$$\Sigma^2 = Var(X(1)+2X(2)) = Var(X(1)) + 4\,Cov(X(1),X(2)) + 4\,Var(X_2)$$
$$= \sigma^2 + 4\sigma^2\cos(1) + 4\sigma^2$$

so

$$\Pr(X(1)+2X(2)\geq 3) = 1-\Pr(N(0,\Sigma^2)\leq 3)$$
$$= 1-\Phi\left(\frac{3-0}{\sqrt{5\sigma^2+4\sigma^2\cos(1)}}\right).$$