Chapter 9 in Miller and Childers

DEF A discrete time discrete valued random process $\{X_n\}_{n=0}^{+\infty}$ is called a <u>Markov chain</u> (and has the Markov property) if for all $x_0, \dots, x_{n+1}$ and n

$$P(X_{n+1}=x_{n+1} \mid X_n=x_n, \dots, X_0=x_0) = P(X_{n+1}=x_{n+1} \mid X_n=x_n)$$

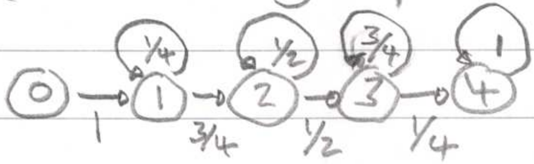
Markov chains are the discrete time special case of more general Markov processes. We study these processes only for discrete time and discrete values.

We assume without loss that our Markov chains are integer $\mathbb{Z}$-valued as they can otherwise be coded to be so.

DEF The <u>transition probabilities</u> $p_{ij} = P(X_{n+1}=j \mid X_n=i)$ are elements of <u>the transition matrix</u> $P = (p_{ij})$.

We always assume so called <u>time homogeneity</u> that p_{ij} does not depend on n.

<u>Example</u> (Kid collecting super heroe figures at fast food restaurant)

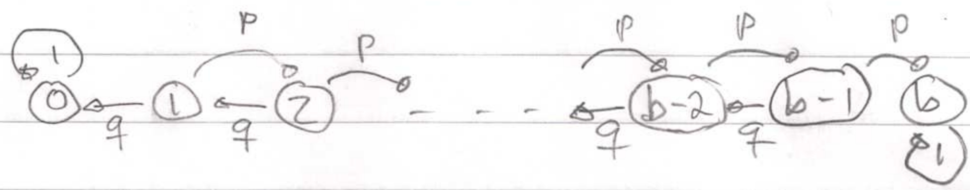


$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 3/4 & 1/4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

X_n = number of different super heroe figures out of four possible collected after n restaurant visits

Example (Random walk) $X_0 = 0$ and $X_n = \sum_{i=1}^{n} Y_i$ where $\{Y_i\}_{i=1}^{+\infty}$ are independent identically distributed with $P(Y_i = 1) = p$ and $P(Y_i = -1) = 1 - p = q$.
The transition matrix is $p_{i,i+1} = p$, $p_{i,i-1} = q$ and $p_{ij} = 0$ for $j \notin \{i-1, i+1\}$.

Example (Gamblers ruin problem) Gambler initially has $X_0 = d$ dollars while house has $b-d$ dollars



$\boxed{\text{DEF}}$ The distribution at time n row matrix $\Pi(n)$ has elements $\Pi(n)_j = P(X_n = j)$.

$\boxed{\text{DEF}}$ The n-step transition probabilities $p_{ij}^{(n)} = P(X_{m+n} = j \mid X_m = i)$ are elements of the n-step transition matrix $P^{(n)} = (p_{ij}^{(n)})$.

$\boxed{\text{THM}}$ (Chapman-Kolmogorov) $\quad P^{(n)} = P^n$

Proof $(P^{(n+1)})_{ij} = P(X_{m+n+1} = j \mid X_m = i, \ldots) = \dfrac{P(X_{m+n+1} = j, X_m = i, \ldots)}{P(X_m = i, \ldots)}$

$$= \sum_{\text{all } k} \frac{P(X_{m+n+1} = j, X_{m+1} = k, X_m = i, \ldots)}{P(X_{m+1} = k, X_m = i, \ldots)} \; \frac{P(X_{m+1} = k, X_m = i, \ldots)}{P(X_m = i, \ldots)}$$

$$= \sum_{\text{all } k} (P^{(n)})_{kj} \, p_{ik} \Rightarrow P^{(n+1)} = P\,P^{(n)} = P\,P\,P^{(n-1)} = \ldots = P^n \quad \#$$

The Markov property is equivalent with the requirement that for all $i_0 < i_1 < \ldots < i_{n+1}$, $x_0, \ldots, x_{n+1}$ and n
$P(X_{i_{n+1}} = x_{n+1} \mid X_{i_n} = x_n, \ldots, X_{i_0} = x_0) = P(X_{i_{n+1}} = x_{n+1} \mid X_{i_n} = x_n)$.

Let us just check this for $i_0=0,\dots,i_n=n,\ i_{n+1}=n+2$:

$$P(X_{n+2}=y \mid X_n=x_n,\dots,X_0=x_0)=\frac{P(X_{n+2}=y,\ X_n=x_n,\dots,X_0=x_0)}{P(X_n=x_n,\dots,X_0=x_0)}$$

$$=\sum_{\text{all } x}\frac{P(X_{n+2}=y,X_{n+1}=x,X_n=x_n,\dots,X_0=x_0)}{P(X_{n+1}=x,X_n=x_n,\dots,X_0=x_0)}\ \frac{P(X_{n+1}=x,X_n=x_n,\dots,X_0=x_0)}{P(X_n=x_n,\dots,X_0=x_0)}$$

$$=\sum_{\text{all } x}P(X_{n+2}=y\mid X_{n+1}=x,X_n=x_n,\dots,X_0=x_0)\,P(X_{n+1}=x\mid X_n=x_n,\dots,X_0=x_0)$$

$$=\sum_{\text{all } x}p_{xy}\,p_{x_n x}=(P^{(2)})_{x_n y}$$ (by Chapman Kolmogorov).

$\boxed{\text{THM}}$ $\Pi(m+n)=\Pi(m)P^{(n)}=\Pi(m)P^n$

Proof $\Pi(m+n)_j=P(X_{m+n}=j)=\sum_{\text{all } k}P(X_{m+n}=j,X_m=k)$

$$=\sum_{\text{all } k}\Pi(m)_k(P^{(n)})_{kj}=(\Pi(m)P^{(n)})_j\ \#$$

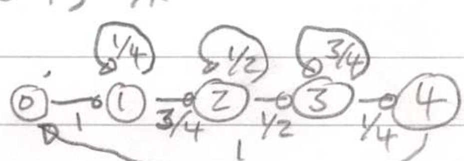
$\boxed{\text{DEF}}$ A row matrix π is called a stationary distribution for a Markov chain if $\pi P=\pi$, $\pi_i\geq 0$ and $\sum_{\text{all } i}\pi_i=1$.

Fact If $\Pi(m)=\pi$ then $\Pi(m+n)=\pi$ for $n\geq 1$.

Proof $\Pi(m+n)=\Pi(m)P^{(n)}=\Pi(m)P^n=\pi P P^{n-1}=\pi P^{n-1}=\dots=\pi$. #

Example For super heroes $\pi=(0\ 0\ 0\ 0\ 1)$ #

Example (Modified super heroes)



$$P=\begin{pmatrix}0&1&0&0&0\\0&1/4&3/4&0&0\\0&0&1/2&1/2&0\\0&0&0&3/4&1/4\\1&0&0&0&0\end{pmatrix}\quad \pi P=\pi \iff \begin{cases}\pi_4=\pi_0\\ \pi_0+1/4\,\pi_1=\pi_1\\ 3/4\,\pi_1+1/2\,\pi_2=\pi_2\\ 1/2\,\pi_2+3/4\,\pi_3=\pi_3\\ 1/4\,\pi_3=\pi_4\end{cases}\iff\begin{array}{l}\pi_1=4/3\,\pi_0\\ \pi_2=2\pi_0\\ \pi_3=4\pi_0\\ \pi_4=\pi_0\end{array}$$

$$\sum_{\text{all } i}\pi_i=\pi_0\left(1+\frac{4}{3}+2+4+1\right)=\frac{28}{3}\Rightarrow \pi=\left(\frac{3}{28}\ \frac{4}{28}\ \frac{6}{28}\ \frac{12}{28}\ \frac{3}{28}\right)$$

DEF The mean time to return to a state (= possible value) i of the Markov chain is defined
$\mu_i = E(\min\{n \geq 1 : X_n = i\} \mid X_0 = i)$

THM For an irreducible aperiodic (see below) Markov chain it holds that π exists if and only if $\mu_i < \infty$ for all i and in that case $\pi_i = 1/\mu_i$ for all i.

Example (Super heroe) As $\mu_0 = 1/\pi_0 = 28/3$ for modified super heroe we have $E(\min\{n \geq 1 : X_n = 4\} \mid X_0 = 0) = 25/3$ for original super heroe. This can also be concluded from using that mean of waiting time distribution is $1/p$ as $1 + 4/3 + 2 + 4 = 25/3$.

DEF j is accessible from i ($i \to j$) if $p_{ij}^{(n)} > 0$ for some n
i and j communicate ($i \leftrightarrow j$) if $i \to j$ and $j \to i$
the chain is irreducible if $i \leftrightarrow j$ for all i and j

Example Random walk and modified super heroe are irreducible while original super heroe is not.

DEF $d(i) = GCD(\{n \geq 1 : p_{ii}^{(n)} > 0\})$ is the period of i.
The chain is aperiodic if $d(i) = 1$ for all i

Example Random walk has $d(i) = 2$ for all i while modified super heroe has $d(i) = 1$ for all i.

DEF i is transient if $P(X_n = i \text{ some } n \geq 1 \mid X_0 = i) < 1$
i is reccurent if $P(X_n = i \text{ some } n \geq 1 \mid X_0 = i) = 1$

$\boxed{\text{THM}}$ i is reccurent if and only if $\sum_{n=1}^{+\infty} p_{ii}^{(n)} = \infty$

$\boxed{\text{THM}}$ For an irreducible chain either all states are reccurent or all states are transient. Further all states have same period and $\mu_i < \infty$ for some i if and only if $\mu_i < \infty$ for all i.

Example For random walk $p_{ii}^{(n)} = \binom{n}{n/2} p^{n/2}(1-p)^{n/2}$ for n even while $p_{ii}^{(n)} = 0$ for n uneven. Using Striling's formula

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n} \quad \text{as } n \to \infty$$

we see that

$$p_{ii}^{(2n)} = \frac{(2n)!\,(p(1-p))^n}{(n!)(n!)} \sim \frac{\sqrt{4\pi n}\,(2n)^{2n} e^{-2n} (p(1-p))^n}{(2\pi n)\, n^{2n}\, e^{-2n}} = \frac{(4 \cdot p(1-p))^n}{\sqrt{\pi n}}$$

so that $\sum_{n=1}^{+\infty} p_{ii}^{(2n)} = \infty$ if and only if $p = 1/2$.