Solutions to Chapter 10 Exercises

Problem 10.11

Let

$$s(t) = \sum_{k=-\infty}^{\infty} s_k \exp(j2\pi f_o t).$$

Then

$$\begin{aligned}
s(t-T) &= \sum_{k=-\infty}^{\infty} s_k \exp(j2\pi f_o (t-T)) \\
R_{X,X}(\tau) &= E\left[\sum_k \sum_m s_k s_m^* \exp(j2\pi k f_o (t-T) \exp(-j2\pi m f_o (t+\tau-T)\right] \\
&= \sum_k \sum_m s_k s_m^* \exp(j2\pi (k-m) f_o t) \exp(-j2\pi m f_o \tau) \\
&\quad E[\exp(j2\pi(k-m)f_o T)] \\
E[\exp(j2\pi(k-m)f_o T)] &= \frac{1}{t_o}\int_0^{t_o} \exp(j2\pi(k-m)f_o u)du \\
&= \begin{cases} 0 & k \neq m \\ 1 & k = m \end{cases} \\
R_{X,X}(\tau) &= \sum_{k=-\infty}^{\infty} |s_k|^2 \exp(-j2\pi k f_o \tau) \\
S_{X,X}(f) &= \sum_{k=-\infty}^{\infty} |s_k|^2 \delta(f - k f_o)
\end{aligned}$$

Hence the process X(t)=s(t-T) has a line spectrum and height of each line is given by the magnitude squared of the Fourier series coefficients.

Problem 10.15

$$R_{X,X}(t_1,t_2) = E[\cos(\omega_c t_1 + B[n_1]\pi/2)\cos(\omega_c t_2 + B[n_2]\pi/2)],$$

where n_1 and n_2 are integers such that $n_1 T \le t_1 < (n_1+1)T$ and $n_2 T \le t_2 < (n_2+1)T/$. For t_1, t_2 such that $n_1 \ne n_2$,

$$R_{X,X}(t_1,t_2) = E[\cos(\omega_c t_1 + B[n_1]\pi/2)]E[\cos(\omega_c t_2 + B[n_2]\pi/2)] = 0,$$

while for t_1, t_2 such that $n_1 = n_2$,

$$\begin{aligned} R_{X,X}(t_1,t_2) &= \frac{1}{2}\cos(\omega_c(t_2-t_1)) + \frac{1}{2}E[\cos(\omega_c(t_2+t_1)+\pi B[n_1])] \\ &= \frac{1}{2}\cos(\omega_c(t_2-t_1)) - \frac{1}{2}\cos(\omega_c(t_2+t_1)) \end{aligned}$$

Since this autocorrelation depends on more than just $t_1 - t_2$, the process is not WSS.
(b) From part (a),

$$R_{X,X}(t,t+\tau) = \begin{cases} 0 & \text{if } t, t+\tau \text{ are in different intervals,} \\ \frac{1}{2}\cos(\omega_c\tau) - \frac{1}{2}\cos(\omega_c(2t+\tau)) & \text{if } t, t+\tau \text{ are in in the same intervals.} \end{cases}$$

Since the process is not WSS we must take time averages.

$$R_{X,X}(\tau) = \langle R_{X,X}(t,t+\tau)\rangle = (1-p(\tau))\langle 0\rangle + p(\tau)\langle \frac{1}{2}\cos(\omega_c\tau) - \frac{1}{2}\cos(\omega_c(2t+\tau))\rangle,$$

where $p(\tau)$ is the fraction of the values of t that lead to t and $t+\tau$ being in the same interval. This function is given by

$$p(\tau) = \begin{cases} 0 & |\tau| > T, \\ 1 - \frac{|\tau|}{T} & |\tau| < T. \end{cases}$$

Therefore,

$$\begin{aligned} p(\tau) &= \text{tri}(t/T) \\ R_{X,X}(\tau) &= \frac{1}{2}\text{tri}(\tau/T)\cos(\omega_c\tau) \\ S_{X,X}(f) &= \frac{1}{2}FT[\text{tri}(\tau/T)] * FT[\cos(\omega_c\tau] \end{aligned}$$

using Table E.1 in Appendix E in the text,

$$\begin{aligned} S_{X,X}(f) &= \frac{1}{4}T\text{sinc}^2(fT) * (\delta(f-f_c) + \delta(f+f_c)) \\ &= \frac{T}{4}(\text{sinc}^2((f-f_c)T)) + \text{sinc}^2((f+f_c)T)) \end{aligned}$$

Problem 10.19

(a) The absolute BW is ∞ since $S(f) > 0$ for all $|f| < \infty$.
(b) The 3dB BW, f_3 satisfies

$$\begin{aligned}
\frac{1}{(1+(f_3/B)^2)^3} &= \frac{1}{2} \\
\Rightarrow f_3 &= B\sqrt{2^{1/3}-1} = 0.5098B.
\end{aligned}$$

(c)

$$\begin{aligned}
\int_{-\infty}^{\infty} f^2 S(f) df &= \int_{-\infty}^{\infty} \frac{f^2}{(1+(f/B)^2)^3} df = B^3 \int_{-\infty}^{\infty} \frac{z^2}{(1+z^2)^3} dz = \frac{\pi}{8} B^3 \\
\int_{-\infty}^{\infty} S(f) df &= \int_{-\infty}^{\infty} \frac{1}{(1+(f/B)^2)^3} df = B \int_{-\infty}^{\infty} \frac{1}{(1+z^2)^3} dz = \frac{3\pi}{8} B \\
B_{rms}^2 &= \frac{\frac{\pi}{8} B^3}{\frac{3\pi}{8} B} = \frac{B^2}{3} \\
B_{rms} &= \frac{B}{\sqrt{3}}.
\end{aligned}$$

Problem 10.23

(a)

$$\begin{aligned} E[\epsilon^2] &= E[(Y[n+1] - a_1 Y[n] - a_2 Y[n-1])^2] \\ &= R_{Y,Y}[0](1 + a_1^2 + a_2^2) - 2a_1(1-a_2)R_{Y,Y}[1] - 2a_2 R_{Y,Y}[2] \end{aligned}$$

(b)

$$\frac{\partial E[\epsilon^2]}{\partial a_1} = 2a_1 R_{Y,Y}[0] - 2(1-a_2)R_{Y,Y}[1] = 0$$

$$\begin{aligned} &\Rightarrow R_{Y,Y}[0]a_1 + R_{Y,Y}[1]a_2 = R_{Y,Y}[1] \\ \frac{\partial E[\epsilon^2]}{\partial a_2} &= 2a_2 R_{Y,Y}[0] - 2R_{Y,Y}[2] + 2a_1 R_{Y,Y}[1] = 0 \\ &\Rightarrow R_{Y,Y}[1]a_1 + R_{Y,Y}[0]a_2 = R_{Y,Y}[2] \\ &\Rightarrow \begin{bmatrix} R_{Y,Y}[0] R_{Y,Y}[1] & \\ R_{Y,Y}[1] & R_{Y,Y}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} R_{Y,Y}[1] \\ R_{Y,Y}[2] \end{bmatrix} \\ \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \frac{1}{R_{Y,Y}^2[0] - R_{Y,Y}^2[1]} \begin{bmatrix} R_{Y,Y}[0]R_{Y,Y}[1] - R_{Y,Y}[1]R_{Y,Y}[2] \\ R_{Y,Y}[0]R_{Y,Y}[2] - R_{Y,Y}^2[1] \end{bmatrix} \end{aligned}$$