Solutions to Chapter 11 Exercises

Problem 11.15

$$
\begin{aligned}
Y(t) &= a(S^2(t) + N^2(t) + 2S(t)N(t)) \\
E[Y(t)] &= aE[S^2(t)] + aE[N^2(t)] + 2aE[S(t)N(t)] \\
&= a(\sigma_S^2 + \sigma_N^2). \\
R_{Y,Y}(\tau) &= a^2E[(S^2(t) + N^2(t) + 2S(t)N(t)) \\
& (S^2(t+\tau) + N^2(t+\tau) + 2S(t+\tau)N(t+\tau))] \\
&= a^2E[S^2(t)S^2(t+\tau)] + a^2E[N^2(t)N^2(t+\tau)] \\
&+ 4a^2E[S(t)S(t+\tau)N(t)N(t+\tau)] + a^2E[S^2(t)N^2(t+\tau)] \\
&+ 2a^2E[S^2(t)S(t+\tau)N(t+\tau)] + a^2E[N^2(t)S^2(t+\tau)] \\
&+ 2a^2E[N^2(t)N(t+\tau)S(t+\tau)] + 2a^2E[S(t)S^2(t+\tau)N(t)] \\
&+ 2a^2E[S(t)N(t)N^2(t+\tau)] \\
&= a^2[R_{S,S}^2(0) + 2R_{S,S}^2(\tau) + R_{N,N}^2(0) + 2R_{N,N}^2(\tau) \\
&+ 4R_{S,S}(\tau)R_{N,N}(\tau) + R_{S,S}(0)R_{N,N}(0) + R_{N,N}(0)R_{S,S}(0)] \\
&= a^2[(R_{S,S}(0) + R_{N,N}(0))^2 + 2(R_{S,S}(\tau) + R_{N,N}(\tau))^2] \\
&= a^2[(\sigma_S^2 + \sigma_N^2)^2 + 2(R_{S,S}(\tau) + R_{N,N}(\tau))^2].
\end{aligned}
$$

In above Calculation we used the following information:

For two Gaussian random process X(t) and Y(t) :

$E\left[X^2(t)Y^2(t+\tau)\right] = R_{XX}[0]R_{YY}[0] + 2R_{XY}^2[\tau]$ If X and Y are correlated

$E\left[X(t)Y(t)Y(t+\tau)\right] = E\left[X(t)\right]E\left[Y(t)Y(t+\tau)\right]$ If X and Y are not correlated

Problem 11.16

For the given PSD's,

$$\begin{aligned} R_{S,S}(\tau) &= F^{-1}[S_{S,S}(f)] = \frac{A^2}{4}(e^{-j2\pi f_c\tau} + e^{j2\pi f_c\tau}) = \frac{A^2}{2}\cos(2\pi f_c\tau) \\ R_{N,N}(\tau) &= F^{-1}[S_{N,N}(f)] = \frac{N_oB}{2}\mathrm{sinc}(B\tau)(e^{-j2\pi f_c\tau} + e^{j2\pi f_c\tau}) \\ &= N_oB\mathrm{sinc}(B\tau)\cos(2\pi f_c\tau) \end{aligned}$$

From these we determine that

$$\begin{aligned} \sigma_S^2 &= R_{S,S}(0) = \frac{A^2}{2}, \\ \sigma_N^2 &= R_{N,N}(0) = N_oB. \end{aligned}$$

The autocorrelation of $Y(t)$ is then

$$\begin{aligned} R_{Y,Y}(\tau) &= a^2[(\sigma_S^2 + \sigma_N^2)^2 + 2(R_{S,S}(\tau) + R_{N,N}(\tau))^2] \\ &= a^2\left[\left(\frac{A^2}{2} + N_oB\right)^2 + 2\left(\frac{A^2}{2} + N_oB\mathrm{sinc}(B\tau)\right)^2\cos^2(2\pi f_c\tau)\right] \\ &= a^2\Bigg[\left(\frac{A^2}{2} + N_oB\right)^2 \\ &+ 2\left(\frac{A^2}{4} + A^2N_oB\mathrm{sinc}(B\tau) + (N_oB)^2\mathrm{sinc}^2(B\tau)\right)(1 + \cos(4\pi f_c\tau))\Bigg] \end{aligned}$$

Taking FT's, the PSD of $Y(t)$ is then

$$S_{Y,Y}(f) = a^2\Bigg[\left(\frac{A^2}{2} + N_oB\right)^2\delta(f) + 2\left(\frac{A^2}{4}\delta(f) + A^2N_o\mathrm{rect}(f/B) + N_o^2B\mathrm{tri}(f/B)\right)$$

$$
\begin{aligned}
& * \left(\delta(f) + \frac{1}{2}\delta(f-2f_c) + \frac{1}{2}\delta(f+2f_c)\right)\Bigg] \\
= \; & a^2\left(\frac{A^2}{2} + A^2N_oB + (N_oB)^2\right)\delta(f) + A^2N_o\mathrm{rect}(f/B) + N_o^2B\mathrm{tri}(f/B) \\
+ \; & \frac{A^2}{4}\left(\delta(f-2f_c)+\delta(f+2f_c)\right) + A^2N_o\left(\mathrm{rect}\left(\frac{f-2f_c}{B}\right) + \mathrm{rect}\left(\frac{f+2f_c}{B}\right)\right) \\
+ \; & N_o^2B\left(\mathrm{tri}\left(\frac{f-2f_c}{B}\right) + \mathrm{tri}\left(\frac{f+2f_c}{B}\right)\right).
\end{aligned}
$$

Problem 11.9

$$S_{Y,Y}(f) = S_{X,X}(f)|H(f)|^2 = \frac{a}{1+(f/f_o)^2}\cdot b^2\left[\mathrm{rect}\left(\frac{f-f_c}{f_2-f_1}\right) + \mathrm{rect}\left(\frac{f+f_c}{f_2-f_1}\right)\right],$$

where $f_c = (f_1+f_2)/2$. Assuming $f_2 - f_1 << f_o$, the input PSD will be approximately constant over the passband of the filter. In that case, the output PSD will be well approximated by

$$
\begin{aligned}
S_{Y,Y}(f) \; & \approx \; S_{X,X}(f_c)|H(f)|^2 = \frac{ab^2}{1+(f_c/f_o)^2}\cdot\left[\mathrm{rect}\left(\frac{f-f_c}{f_2-f_1}\right) + \mathrm{rect}\left(\frac{f+f_c}{f_2-f_1}\right)\right] \\
\Rightarrow \; & = \; \frac{2ab^2(f_2-f_1)}{1+(f_c/f_o)^2}\mathrm{sinc}((f_2-f_1)\tau)\cos(\omega_c\tau).
\end{aligned}
$$

Problem 11.34

$h(t)=s(t_0-t)$. In this case, $s(t_0-t)=s(t)$ so the impulse response of the matched filter is the same as the signal itself.