

Solutions to Chapter 3 Exercises

Problem 3.12

$$f_X(x) = \begin{cases} a^{-bx} & x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Since this is a pdf the following integral should evaluate to 1

$$\begin{aligned} \int_{-\infty}^{+\infty} f_X(x) dx &= 1 \\ \int_{-\infty}^0 f_X(x) dx + \int_0^{+\infty} f_X(x) dx &= 1 \\ \int_{-\infty}^0 a^{-bx} dx + \int_0^{+\infty} 0 dx &= 1 \\ \left. \frac{-a^{-bx}}{b \ln(a)} \right|_{-\infty}^0 &= 1 \\ -\frac{1}{b \ln(a)} + \lim_{x \rightarrow -\infty} \frac{a^{-bx}}{b \ln(a)} &= 1 \end{aligned}$$

Here two cases arise $a > 1$ and $0 < a < 1$. The case $a < 0$ is not possible because then $f_X(x)$ can become imaginary and will no longer be a valid pdf. The case $a = 0$ is also not possible because then the denominator become zero. First considering the case $a > 1$. This evaluates to a finite value only if $b < 0$. And in the case $a < 1$ the limit will be finite only if $b > 0$. In both cases the limit evaluates to 0 and the above equation reduces to

$$\begin{aligned} -\frac{1}{b \ln(a)} &= 1 \\ a &= e^{-1/b} \end{aligned}$$

We can see that this relation satisfies the requirements we laid on a and b earlier. Also using this relation the pdf can be written as:

$$f_X(x) = \left(e^{-1/b} \right)^{bx} = e^x$$

b) The CDF is given by

$$F_X(x) = \int_{-\infty}^x f_X(x) dx = \int_{-\infty}^x e^x dx = e^x$$

$$x = 0 \rightarrow F_X(0) = e^0 = 1 \rightarrow F_X(x) = \begin{cases} e^x & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Problem 3.13

a) Since

$$\int_{-\infty}^{+\infty} f_X(x) dx = 1$$

$$c \int_{-\infty}^{+\infty} e^{-2x} u(x) dx = 1$$

$$c \int_0^{+\infty} e^{-2x} dx = 1$$

$$-\frac{c}{2} e^{-2x} \Big|_0^{+\infty} = 1 \rightarrow c = 2, f_X(x) = 2e^{-2x} u(x)$$

b)

$$\Pr(X > 2) = 2 \int_2^{+\infty} e^{-2x} dx = e^{-4} = 0.0183$$

c)

$$\Pr(X < 3) = 2 \int_0^3 e^{-2x} dx = 1 - e^{-6} = 0.9975$$

d)

$$\begin{aligned} \Pr(X < 3 | X > 2) &= \frac{\Pr(2 < X < 3)}{\Pr(X > 2)} \\ &= \frac{2 \int_2^3 e^{-2x} dx}{e^{-4}} \\ &= \frac{e^{-4} - e^{-6}}{e^{-4}} \\ &= 1 - e^{-2} = 0.8647 \end{aligned}$$

Problem 3.16

$$\begin{aligned}\Pr(|S - 10| > 0.075) &= \int_{9.9}^{10-0.075} f_S(s) ds + \int_{10+0.075}^{10.1} f_S(s) ds \\ &= 2 \int_{9.9}^{9.925} f_S(s) ds \\ &= 2 \int_{9.9}^{9.925} 100(s - 9.9) ds \\ s - 9.9 = u \quad \rightarrow \quad &= 2 \int_0^{0.025} 100 u du \\ &= 100u^2 \Big|_0^{0.025} \\ &= 0.0625\end{aligned}$$

Problem 3.21

$$f_X(x) = ce^{-2x^2-3x-1}$$

a) As usual the integral of the pdf evaluates to 1

$$\begin{aligned} 2x^2 + 3x + 1 &= 2\left(x^2 + \frac{3}{2}x + \frac{1}{2}\right) = 2\left(x^2 + \frac{3}{2}x + \frac{9}{16} - \frac{1}{16}\right) \\ &= 2\left(x^2 + \frac{3}{2}x + \frac{9}{16}\right) - \frac{1}{8} = 2\left(x + \frac{3}{4}\right)^2 - \frac{1}{8} \end{aligned}$$

$$\int_{-\infty}^{+\infty} f_X(x) dx = 1 \rightarrow \int_{-\infty}^{+\infty} ce^{-(2x^2+3x+1)} dx = 1$$

$$\int_{-\infty}^{+\infty} ce^{-2\left(x+\frac{3}{4}\right)^2 + \frac{1}{8}} dx = 1$$

$$ce^{\frac{1}{8}} \int_{-\infty}^{+\infty} e^{-2\left(x+\frac{3}{4}\right)^2} dx = 1$$

$$ce^{\frac{1}{8}} \sqrt{\frac{\pi}{2}} \int_{-\infty}^{+\infty} \frac{1}{\frac{1}{2}\sqrt{2\pi}} e^{-\frac{\left(x+\frac{3}{4}\right)^2}{2\left(\frac{1}{2}\right)^2}} dx = 1$$

$$ce^{\frac{1}{8}} \sqrt{\frac{\pi}{2}} = 1$$

$$c = e^{-\frac{1}{8}} \sqrt{\frac{2}{\pi}} = 0.7041$$

b) The previous pdf can be rewritten in the form of a standard Gaussian pdf as follows

$$f_X(x) = ce^{-2x^2-3x-1} = e^{-\frac{1}{8}} \sqrt{\frac{2}{\pi}} e^{-2\left(x+\frac{3}{4}\right)^2 + \frac{1}{8}}$$

$$= \sqrt{\frac{2}{\pi}} e^{-2\left(x+\frac{3}{4}\right)^2}$$

This is in the form of standard Gaussian pdf given by $f_X(x) = \sqrt{\frac{1}{2\pi\sigma^2}} \exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\}$ and we can easily identify the mean m and the standard deviation σ as

$$m = -\frac{3}{4}, \quad \sigma^2 = \frac{1}{4} \rightarrow \sigma = \frac{1}{2}$$

Problem 3.36

$$\Pr(M = 0) = P_0$$

$$\Pr(M = 1) = P_1$$

$$f_{X|M=0}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}$$

$$f_{X|M=1}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-1)^2}{2\sigma^2}\right\}$$

$$f_X(x) = f_{X|M=0}(x)\Pr(M=0) + f_{X|M=1}(x)\Pr(M=1)$$

$$= f_{X|M=0}(x)P_0 + f_{X|M=1}(x)P_1$$

$$\Pr(M=0|X=x) = \frac{f_{X|M=0}(x)\Pr(M=0)}{f_X(x)}$$

$$= \frac{f_{X|M=0}(x)P_0}{f_X(x)}$$

$$= \frac{\frac{P_0}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}}{\frac{P_0}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} + \frac{P_1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-1)^2}{2\sigma^2}\right\}}$$

$$= \frac{1}{1 + \frac{P_1}{P_0} \exp\left\{\frac{2x-1}{2\sigma^2}\right\}}$$

a)

$$P_0 = P_1 = 0.5, \sigma^2 = 1$$
$$\rightarrow \Pr(M = 0|X = x) = \frac{1}{1 + \exp\{x - 0.5\}}$$

$$P_0 = P_1 = 0.5, \sigma^2 = 5$$
$$\rightarrow \Pr(M = 0|X = x) = \frac{1}{1 + \exp\left\{\frac{2x-1}{10}\right\}}$$

b)

$$P_0 = 0.25, P_1 = 0.75, \sigma^2 = 1$$
$$\rightarrow \Pr(M = 0|X = x) = \frac{1}{1 + 3 \exp\{x - 0.5\}}$$

$$P_0 = 0.25, P_1 = 0.75, \sigma^2 = 5$$
$$\rightarrow \Pr(M = 0|X = x) = \frac{1}{1 + 3 \exp\left\{\frac{2x-1}{10}\right\}}$$

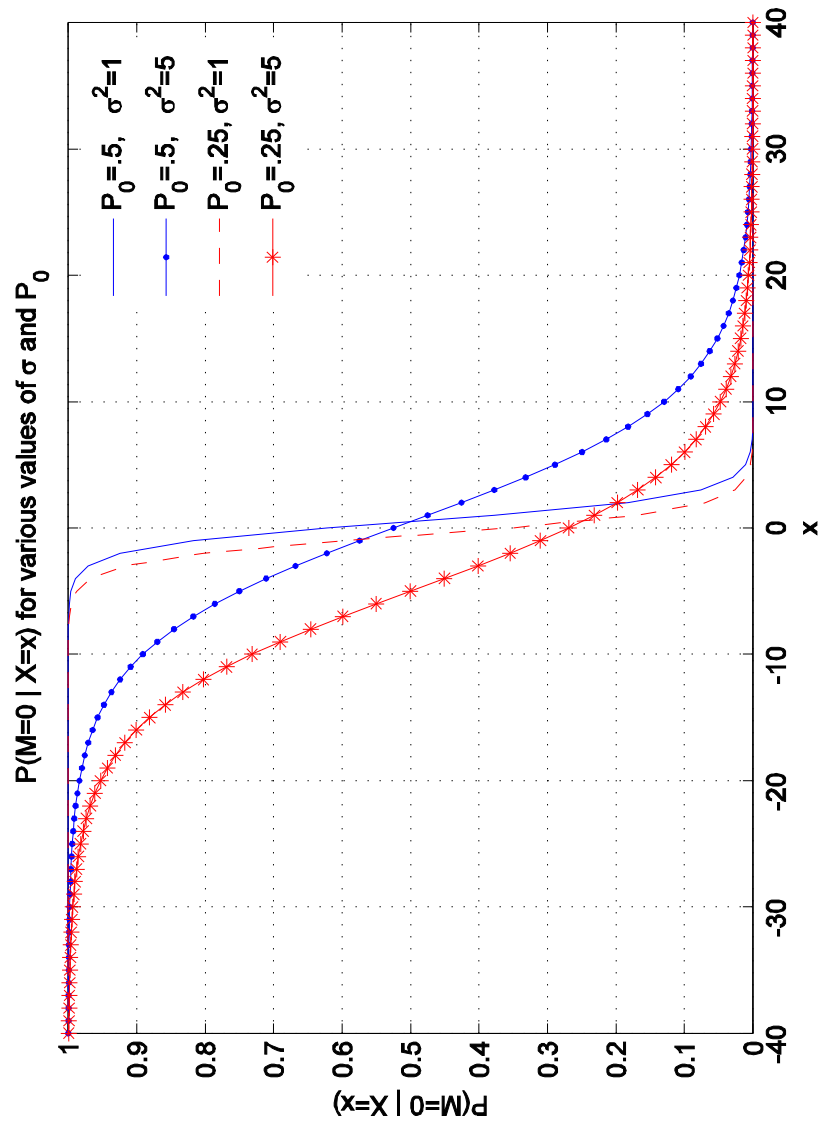


Figure 1

Problem 3.37

a) We decide 0 if $\Pr(M = 0 | X = x) \geq 0.9$. Using the achieved result in problem 3.16

for $P_0 = P_1 = 0.5, \sigma^2 = 1$

$$\begin{aligned}\Pr(M = 0|X = x) &= \frac{1}{1 + \exp\{x - 0.5\}} \geq 0.9 \\ &\rightarrow \exp\{x - 0.5\} \leq \frac{1}{9} \\ &\rightarrow x \leq -1.6972\end{aligned}$$

In the same way, we decide 1 if $\Pr(M = 1|X = x) \geq 0.9$.

$$\begin{aligned}\Pr(M = 1|X = x) &= \frac{f_{X|M=1}(x)\Pr(M = 1)}{f_X(x)} \\ &= \frac{f_{X|M=1}(x)P_1}{f_X(x)} \\ &= \frac{\frac{P_1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-1)^2}{2\sigma^2}\right\}}{\frac{P_0}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} + \frac{P_1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-1)^2}{2\sigma^2}\right\}} \\ &= \frac{1}{1 + \frac{P_0}{P_1} \exp\left\{\frac{1-2x}{2\sigma^2}\right\}}\end{aligned}$$

for $P_0 = P_1 = 0.5, \sigma^2 = 1$

$$\begin{aligned}\Pr(M = 1|X = x) &= \frac{1}{1 + \exp\{0.5 - x\}} \geq 0.9 \\ &\rightarrow \exp\{0.5 - x\} \leq \frac{1}{9} \\ &\rightarrow x \geq 2.6972\end{aligned}$$

In the end we can formulize the decision rules as follows

$$\begin{cases} 0 & x \leq -1.6972 \\ 1 & x \geq 2.6972 \\ Erased & -1.6972 < x < 2.6972 \end{cases}$$

b)

$$\begin{aligned}
P(\text{Erased}) &= P(-1.6972 < x < 2.6972) \\
&= P(-1.6972 < x < 2.6972 | M = 0)P_0 + P(-1.6972 < x < 2.6972 | M = 1)P_1 \\
&= \frac{P(-1.6972 < x < 2.6972 | M = 0)}{2} + \frac{P(-1.6972 < x < 2.6972 | M = 1)}{2} \\
\left\{ \begin{array}{l} x | M = 0 : N(0,1) \\ x | M = 1 : N(1,1) \\ F_x(x) = 1 - Q\left(\frac{x-m}{\sigma}\right) \end{array} \right. &\rightarrow P(\text{Erased}) = \frac{Q(-1.6972) - Q(2.6972)}{2} + \frac{Q(-1.6972 - 1) - Q(2.6972 - 1)}{2} \\
&= \frac{1 - Q(1.6972) - Q(2.6972)}{2} + \frac{1 - Q(2.6972) - Q(1.6972)}{2} \\
&= 1 - Q(1.6972) - Q(2.6972) \\
&= 0.95196
\end{aligned}$$

c)

$$\begin{aligned}
P(\text{Error}) &= P(\text{Error} | M = 0)P_0 + P(\text{Error} | M = 1)P_1 \\
&= \frac{P(x \geq 2.6972 | M = 0)}{2} + \frac{P(x \leq -1.6972 | M = 1)}{2} \\
&= \frac{Q(2.6972)}{2} + \frac{1 - Q(-1.6972 - 1)}{2} \\
&= Q(2.6972) \\
&= 0.00347
\end{aligned}$$