# Solutions to Chapter 4 Exercises

### Problem 4.29

We know the pdf of a distribution can be written as sum of the conditional pdfs.

$$\begin{aligned} f_X(x) &= \sum_{i=1}^{n} f_{X|A_i}(x) Pr(A_i) \\ E[X] &= \int_{-\infty}^{\infty} x f_X(x)\, dx \\ &= \int_{-\infty}^{\infty} x \sum_{i=1}^{n} f_{X|A_i}(x) Pr(A_i)\, dx \end{aligned}$$

We can interchange the operations of the integration and summation as they are linear and rewrite the above equation as

$$\begin{aligned} E[X] &= \sum_{i=1}^{n} \left( \int_{-\infty}^{\infty} x f_{X|A_i}(x) Pr(A_i)\, dx \right) \\ E[X] &= \sum_{i=1}^{n} \left( Pr(A_i) \int_{-\infty}^{\infty} x f_{X|A_i}(x)\, dx \right) \\ E[X] &= \sum_{i=1}^{n} Pr(A_i) E[X|A_i] \end{aligned}$$

## Problem 4.25

$$f_\Theta(\theta) = \frac{1}{2\pi}$$

(a) $Y = \sin\theta$. This equation has two roots. At $\theta$ and $\pi - \theta$

$$\begin{aligned}
f_Y(y) &= \sum_{\theta_i} f_\Theta(\theta) \left|\frac{d\theta}{dy}\right|_{\theta=\theta_i} \\
&= \left|\frac{1}{2\pi\cos\theta}\right|_{\theta=\theta} + \left|\frac{1}{2\pi\cos\theta}\right|_{\theta=\pi-\theta} \\
&= \frac{1}{\pi\cos\theta} \\
&= \frac{1}{\pi\sqrt{1-y^2}}
\end{aligned}$$

(b) $Z = \cos\theta$. This equation has two roots. At $\theta$ and $2\pi - \theta$

$$\begin{aligned}
f_Z(z) &= \sum_{\theta_i} f_\Theta(\theta) \left|\frac{d\theta}{dy}\right|_{\theta=\theta_i} \\
&= \left|\frac{1}{2\pi\sin\theta}\right|_{\theta=\theta} + \left|\frac{1}{2\pi\sin\theta}\right|_{\theta=2\pi-\theta} \\
&= \frac{1}{\pi\sin\theta} \\
&= \frac{1}{\pi\sqrt{1-y^2}}
\end{aligned}$$

(c) $W = \tan\theta$. This equation has two roots. At $\theta$ and $\pi + \theta$

$$
\begin{aligned}
f_W(w) &= \sum_{\theta_i} f_\Theta(\theta)\left|\frac{d\theta}{dy}\right|_{\theta=\theta_i} \\
&= \left|\frac{1}{2\pi\sec^2\theta}\right|_{\theta=\theta} + \left|\frac{1}{2\pi\sec^2\theta}\right|_{\theta=\pi+\theta} \\
&= \frac{1}{\pi\sec^2\theta} \\
&= \frac{1}{\pi(\tan^2\theta+1)} \\
&= \frac{1}{\pi(w^2+1)}
\end{aligned}
$$

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## Problem 4.38

$$
\begin{cases}
P_Y(y=-2) = P_X(x<-1) = Q\left(\frac{1}{\sigma_X}\right) & (1) \\
P_Y(-2 \le y \le 2) = P_X(-1 \le x \le 1) & (2) \\
P_Y(y=2) = P_X(x>1) = Q\left(\frac{1}{\sigma_X}\right) & (3)
\end{cases}
$$

For middle part (2), Y=2X, so we can write:

$$
\begin{aligned}
f_Y(y) &= f_X(x)\left\|\frac{dx}{dy}\right\|_{x=\frac{1}{2}y} \\
&= \frac{1}{2} f_X\left(\frac{1}{2}y\right) \\
&= \frac{1}{2\sqrt{2\pi\sigma_X^2}}\exp\left(\frac{-\left(\frac{1}{2}y\right)^2}{2\sigma_X^2}\right) \\
&= \frac{1}{2\sqrt{2\pi\sigma_X^2}}\exp\left(\frac{-y^2}{8\sigma_X^2}\right)
\end{aligned}
$$

Finally we get:

$$Y \in [-2, 2]$$

$$f_Y(y) = Q(\frac{1}{\sigma_X})\delta(y+2) + \frac{1}{2\sqrt{2\pi\sigma_X^2}} \exp\left(\frac{-y^2}{8\sigma_X^2}\right) + Q(\frac{1}{\sigma_X})\delta(y-2)$$

$$= Q(\frac{1}{\sigma_X})[\delta(y+2) + \delta(y-2)] + \frac{1}{2\sqrt{2\pi\sigma_X^2}} \exp\left(\frac{-y^2}{8\sigma_X^2}\right)$$

## ***Attention please!***

**For Pr(Y=±2), we have two separate parts. One part is included in (1) and (3) and second part in (2), therefore the step functions which we talked about them in exercise session, are not necessary as far as we know Y is in [-2,2].**

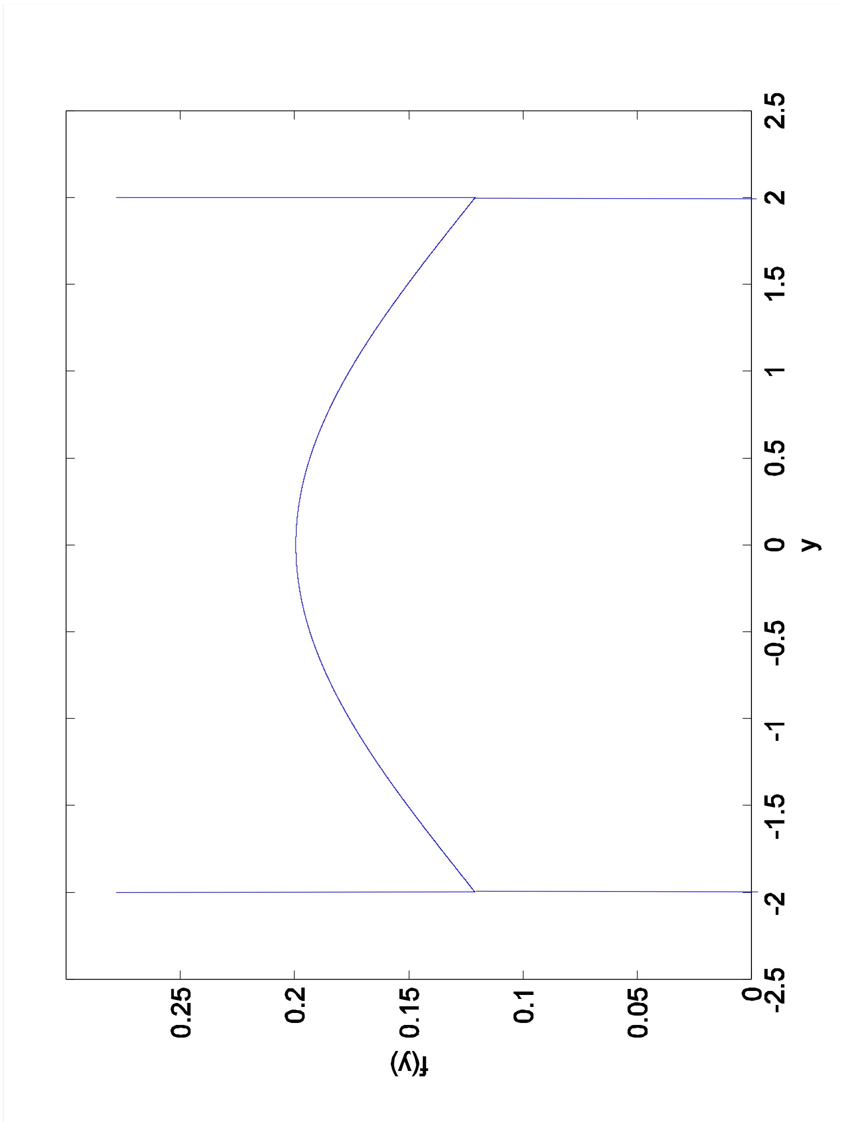


**Figure 1**

## Problem 4.40

$a) X \in [-0.5, 0.5)$

$b)$ X is unform over $[-0.5, 0.5)$

$$f_X(x) = \frac{1}{0.5 - (-0.5)} = 1$$

$$c) E\left[X^2\right] = \int_{-\infty}^{+\infty} x^2 f_X(x) dx = \int_{-0.5}^{+0.5} x^2 dx = \frac{x^3}{3}\Big|_{-0.5}^{0.5} = \frac{1}{12}$$

**Problem 4.53**

(a) Characteristic Function

$$\begin{aligned}
f_X(x) &= \frac{1}{2b} exp(-\frac{|x|}{b}) \\
\Phi_X(\omega) &= \int_{-\infty}^{\infty} \frac{1}{2b} e^{-\frac{|x|}{b}} e^{j\omega x} \\
&= \int_{-\infty}^{\infty} \frac{1}{2b} e^{-\frac{|x|}{b}} e^{j\omega x} \\
&= \int_{-\infty}^{0} \frac{1}{2b} e^{\frac{x}{b}} e^{j\omega x} + \int_{0}^{\infty} \frac{1}{2b} e^{-\frac{x}{b}} e^{j\omega x} \\
&= \frac{1}{2b} \left( \left[ \frac{e^{\frac{x}{b}+j\omega x}}{\frac{1}{b}+j\omega} \right] \Bigg|_{x=-\infty}^{x=0} + \left[ \frac{e^{-\frac{x}{b}+j\omega x}}{-\frac{1}{b}+j\omega} \right] \Bigg|_{x=0}^{x=\infty} \right) \\
&= \frac{1}{2b} \left( \frac{1}{\frac{1}{b}+j\omega} - \frac{1}{-\frac{1}{b}+j\omega} \right) \\
&= \frac{1}{2b} \left( \frac{1}{\frac{1}{b}+j\omega} + \frac{1}{\frac{1}{b}-j\omega} \right) \\
&= \frac{1}{1+b^2\omega^2}
\end{aligned}$$

(b) Taylor Series Expansion of $\Phi_X(\omega)$.

$$\begin{aligned}
\Phi_X(\omega) &= \frac{1}{1+b^2\omega^2} \\
&= \sum_{k=0}^{\infty} (-1)^k (b\omega)^{2k}
\end{aligned}$$

(c) $k^{th}$ Moment of $X$.

$$E[X^k] = (-j)^k \frac{d^k \Phi_X(\omega)}{d\omega^k} \bigg|_{\omega=0}$$

$$\Phi_X(\omega) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{d^k \Phi_X(\omega)}{d\omega^k} \bigg|_{\omega=0} \right) \omega^k = \sum_{m=0}^{\infty} (-1)^m b^{2m} \omega^{2m}$$

Since there are no odd powers in the Taylor series expansion of $\Phi_X(\omega)$, all odd moments of $X$ are zero. For even values of $k$, we note from the above expressions that

$$\frac{1}{(2k)!} \left( \frac{d^{2k} \Phi_X(\omega)}{d\omega^{2k}} \bigg|_{\omega=0} \right) = (-1)^k b^{2k}$$

$$\Rightarrow \frac{d^k \Phi_X(\omega)}{d\omega^k} \bigg|_{\omega=0} = (k!) j^k b^k$$

$$\Rightarrow E[X^k] = (k!) b^k.$$