# Solutions to Chapter 5 Exercises

**Problem 5.51**

$$\begin{aligned} X \sim N(1,4) \\ Y \sim N(-2,9) \\ Z &= 2X - 3Y - 5 \\ \rho_{X,Y} &= \frac{1}{3} \\ \Rightarrow Cov(X,Y) &= \frac{3 \times 2}{3} = 2 \end{aligned}$$

We will make use of the fact that a linear transformation of the Gaussian Random variables results in a Gaussian Random variable.

$$E[Z] = E[2X - 3Y - 5] = 2(1) - 3(-2) - 5 = 3$$

$$\begin{aligned} Var(Z) &= E[(Z-3)^2] = E[(2X - 3Y - 8)^2] \\ &= E[(2X - 2 - 3Y - 6)^2] \\ &= E[4(X-1)^2 + 9(Y+2)^2 - 12(X-1)(Y+2)] \\ &= 4Var(X) + 9Var(Y) - 12Cov(X,Y) \\ &= 4(4) + 9(9) - 12(2) = 73 \end{aligned}$$

Hence $Z$ is a Gaussian distributed as follows

$$Z \sim N(3, 73)$$

## Problem 5.56

Since the transformation is linear and $X$ and $Y$ are jointly Gaussian, $U$ and $V$ will be jointly Gaussian with

$$\begin{aligned}
E[U] &= E[X]\cos(\theta) - E[Y]\sin(\theta) = 0 \\
E[V] &= E[X]\sin(\theta) + E[Y]\cos(\theta) = 0 \\
Var(U) &= E[U^2] = E[X^2]\cos^2(\theta) + E[Y^2]\sin^2(\theta) - 2E[XY]\cos(\theta)\sin(\theta) \\
&= \cos^2(\theta) + \sin^2(\theta) = 1 \\
Var(V) &= E[V^2] = E[X^2]\sin^2(\theta) + E[Y^2]\cos^2(\theta) + 2E[XY]\cos(\theta)\sin(\theta) \\
&= \cos^2(\theta) + \sin^2(\theta) = 1 \\
Cov(U,V) &= E[UV] = E[X^2]\cos(\theta)\sin(\theta) - E[Y^2]\cos(\theta)\sin(\theta) + E[XY](\cos^2(\theta) - \sin^2(\theta)) \\
&= \cos(\theta)\sin(\theta) - \cos(\theta)\sin(\theta) = 0
\end{aligned}$$

Hence, $U$ and $V$ are independent standard Normal random variables.