Solutions to Chapter 8 Exercises

Problem 8.9

(a)

$R_{Y,Y}[n_1, n_2] = E[(X[n_1]+c)(X[n_2]+c)] = R_{X,X}[n_1, n_2]+c\mu_X[n_1]+c\mu_X[n_2]+c^2$

Since $X[n]$ is WSS, $\mu_X[n] = \mu_X$ and $R_{X,X}[n_1, n_2] = R_{X,X}[n_2 - n_1]$.

$$\Rightarrow R_{Y,Y}[n_1, n_2] = R_{X,X}[n_2 - n_1] + 2c\mu_X + c^2.$$

(b)

$$\begin{aligned} E[X[n_1]Y[n_2]] &= E[X[n_1](X[n_2] + c)] = R_{X,X}[n_2 - n_1] + c\mu_X. \\ E[X[n_1]]E[Y[n_2]] &= \mu_X(\mu_X + c) = \mu_X^2 + c\mu_X. \end{aligned}$$

The processes are not orthogonal (since $R_{X,Y}[n_1, n_2] \neq 0$).
The processes are not uncorrelated (since $R_{X,Y}[n_1, n_2] \neq \mu_x\mu_Y$).
The processes are not independent (since not uncorrelated and since $Y[n] = X[n] + c$).

Problem 8.11

(a)

$$\mu_X(t) = \mu_A \cos(\omega t) + \mu_B \sin(\omega t) = 0.$$

(b)

$$\begin{aligned} R_{X,X}(t_1, t_2) &= E[A^2]\cos(\omega t_1)\cos(\omega t_2) + E[B^2]\sin(\omega t_1)\sin(\omega t_2) \\ &+ E[AB]\cos(\omega t_1)\sin(\omega t_2) + E[AB]\sin(\omega t_1)\cos(\omega t_2) \\ &= \frac{E[A^2] + E[B^2]}{2}\cos(\omega(t_2 - t_1)) + \frac{E[A^2] - E[B^2]}{2}\cos(\omega(t_1 + t_2)). \end{aligned}$$

(c) $X(t)$ will be WSS if $E[A^2] = E[B^2] \Rightarrow \sigma_A^2 = \sigma_B^2$.

Problem 8.15

(a) Since T is uniformly distributed over one period of $s(t)$, for any time instant t, $X(t) = s(t-T)$ will be equally likely to take on any of the values in one period of $s(t)$. Since $s(t)$ is 1 half of the time and -1 half of the time, we get

$$\Pr(X(t) = 1) = \Pr(X(t) = -1) = \frac{1}{2}.$$

(b)

$$E[X(t)] = (1) \cdot \Pr(X(t) = 1) + (-1) \cdot \Pr(X(t) = -1) = 0.$$

This can also be seen in an alternative manner:

$$E[X(t)] = E[s(t-T)] = \int s(t-u) f_T(u) du = \int_0^1 s(t-u) du.$$

Since the integral is over one period of $s(t)$, $E[X(t)]$ is just the d.c. value (time average) of $s(t)$ which is zero.

(c)

$$\begin{aligned} R_{X,X}(t_1, t_2) &= E[s(t_1 - T)s(t_2 - T)] \\ &= \int_0^1 s(t_1 - u)s(t_2 - u)du \\ &= \int_0^1 s(v)s(v + t_2 - t_1)dv \\ &= s(t) * s(-t)\Big|_{t = t_2 - t_1}. \end{aligned}$$

This is the time correlation of a square wave with itself which will result in the periodic triangle wave shown in Figure 1.

(d) The process is WSS.

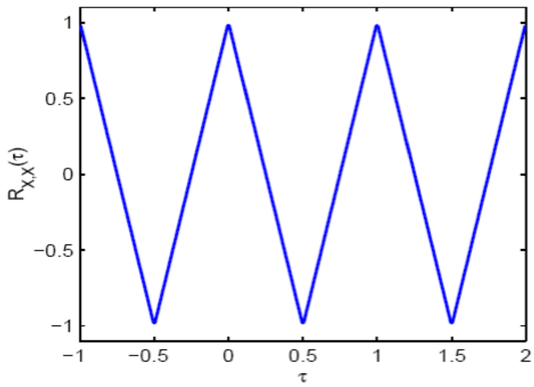

Figure 1: Autocorrelation function for process of Exercise 8.11

Problem 8.18

(a)

$$f_X(x;t) = \frac{f_A(a)}{\left|\frac{dX}{dA}\right|}\Bigg|_{A=-\frac{1}{t}\ln(x)} = \frac{f_A(-\frac{1}{t}\ln(x))}{tx}$$

(b)

$$E[X(t)] = E[e^{-At}] = \int_0^\infty e^{-at}e^{-a}da = \frac{1}{1+t}.$$

$$R_{X,X}(t_1,t_2) = E[X(t_1)X(t_2)] = E[e^{-A(t_1+t_2)}] = \frac{1}{1+t_1+t_2}.$$

The process is not WSS.

Problem 8.33

(a) Consider a time instant, t, such that $0 < t < t_o$.

$$\begin{aligned}
\Pr(X(t)=1|X(t_o)=1) &= \frac{\Pr(X(t_o)=1|X(t)=1)\Pr(X(t)=1)}{\Pr(X(t_o)=1)} \\
&= \frac{\lambda t e^{-\lambda t}}{\lambda t_o e^{-\lambda t_o}} \Pr(\text{no arrivals in } (0,t)) \\
&= \frac{t}{t_o} \exp(-\lambda(t-t_o)) \exp(-\lambda(t_o-t)) \\
&= \frac{t}{t_o}.
\end{aligned}$$

Let S_1 be the arrival time of the first arrival. Then $\{X(t)=1\} \Leftrightarrow \{S_1 \le t\}$. Hence, given that there is one arrival in $(0,t_o)$, that is $X(t_o)=1$,

$$\Pr(X(t)=1|X(t_o)=1) = \Pr(S_1 \le t|X(t_o)=1) = F_{S_1}(t|X(t_o)=1) = \frac{t}{t_o}$$

$$\Rightarrow f_{S_1}(t|X(t_o)=1) = \frac{1}{t_o}, \qquad 0 \le t < t_o.$$

(b) Let $0 \le t_1 \le t_2 \le t_o$. Also define

$$\begin{aligned}
S_1 &= \text{arrival time of first arrival,} \\
S_2 &= \text{arrival time of second arrival.}
\end{aligned}$$

The joint distribution of the two arrival times is found according to:

$$\begin{aligned}
f_{S_1,S_2}(t_1,t_2|X(t_o)=2) &= f_{S_1|S_2}(t_1|S_2=t_2, X(t_o)=2) f_{S_2}(t_2|X(t_o)=2) \\
&= f_{S_1|S_2}(t_1|S_2=t_2) f_{S_2}(t_2|X(t_o)=2)
\end{aligned}$$

To find $f_{S_2}(t_2)$, proceed as in part (a).

$$\begin{aligned}
F_{S_2}(t_2|X(t_o)=2) &= \Pr(X(t_2)=2|X(t_o)=2) \\
&= \Pr(X(t_o)=2|X(t_2)=2)\frac{\Pr(X(t_2)=2)}{\Pr(X(t_o)=2)} \\
&= \exp(-\lambda(t_o-t_2))\frac{\frac{(\lambda t_2)^2}{2}e^{-\lambda t_2}}{\frac{(\lambda t_0)^2}{2}e^{-\lambda t_o}} \\
&= \left(\frac{t_2}{t_o}\right)^2. \\
\Rightarrow f_{S_2}(t_2|X(t_o)=2) &= \frac{2t_2}{t_o^2}, \qquad 0 \le t_2 \le t_o.
\end{aligned}$$

Given $S_2 = t_2$ there is one arrival between 0 and t_2. From the results of part (a), we know S_1 is uniform over $(0, t_2)$ given $S_2 = t_2$. Therefore

$$f_{S_1|S_2}(t_1|t_2) = \frac{1}{t_2}, \qquad 0 \le t_1 \le t_2.$$

Putting the two previous results together we get

$$\begin{aligned} f_{S_1,S_2}(t_1, t_2|X(t_o) = 2) &= f_{S_1|S_2}(t_1|S_2 = t_2) f_{S_2}(t_2|X(t_o) = 2) \\ &= \frac{2t_2}{t_o^2} \cdot \frac{1}{t_2} \\ &= \frac{2}{t_o^2}, \qquad 0 \le t_1 \le t_2 \le t_o. \end{aligned}$$

The two arrival times S_1 and S_2 are uniformly distributed over $0 \le t_1 \le t_2 \le t_o$.

In General we can write:

$$f_{s_1, s_2, \ldots, s_n}\left(t_1, t_2, \ldots, t_n \middle| X(t_0) = n\right) = \frac{n!}{t_0^n}$$

Problem 8.34

$$\begin{aligned} \Pr(N(t) = k|N(t+\tau) = m) &= \Pr(N(t+\tau) = m|N(t) = k) \frac{\Pr(N(t) = k)}{\Pr(N(t+\tau) = m)} \\ &= \frac{\frac{(\lambda\tau)^{m-k}}{(m-k)!} e^{-\lambda\tau} \frac{(\lambda t)^k}{k!} e^{-\lambda t}}{\frac{(\lambda(t+\tau))^m}{m!} \exp(-\lambda(t+\tau))} \\ &= \binom{m}{k} \frac{t^k \tau^{m-k}}{(t+\tau)^m}. \end{aligned}$$

Problem 8.37

$$\Pr(N(t) < 10) = \sum_{k=0}^{9} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

(a)

$$\lambda = 0.1, t = 10 \Rightarrow Pr(N(t) < 10) = \sum_{k=0}^{9} \frac{(1)^k}{k!} e^{-1} \approx 1.$$

(b)

$$\lambda = 10, t = 10 \Rightarrow Pr(N(t) < 10) = \sum_{k=0}^{9} \frac{(100)^k}{k!} e^{-100} \approx 0.$$

(c)

$$\begin{aligned}
\Pr(\text{1 call in 10 minutes}) &= 1 \cdot e^{-1} = 0.3679. \\
\Pr(\text{2 calls in 10 minutes}) &= \frac{1^2}{2!} \cdot e^{-1} = 0.1839. \\
\Pr(\text{1 call, 2 calls}) &= \Pr(\text{1 call}) \Pr(\text{2 calls}) \\
&= \frac{1^3}{2!1!} e^{-2} = 0.0677.
\end{aligned}$$