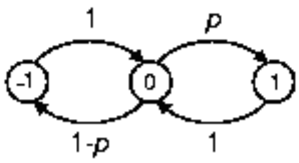
Solutions to chapter 9 Exercises

Problem 9.22



a) The states of the system can be values of the voltage between switches. Hence there are three states, namely -1, 0, and +1. With this representation, the process is a random walk with reflecting boundaries. The Corresponding state transition matrix is

$$P = \begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix}$$

b)

$$P^2 = \begin{bmatrix} q & 0 & p \\ 0 & 1 & 0 \\ q & 0 & p \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} q & 0 & p \\ 0 & 1 & 0 \\ q & 0 & p \end{bmatrix}$$

We know that a *recurrent state* i is a *periodic state* if $p_{i,i}^{(n)} > 0$ for n>1. And also state i is recurrent if $P_{i,i}^{(n)} = 1$ for some $n \geq 1$. In above states, the only recurrent state is state 0 and two other states are transient. Therefore they can't be periodic and we have just to determine if state 0 is periodic and if so what its period is. For this process $p_{0,0}^{(n)} > 0$ only if n is even. Therefore, the process is periodic with period 2.

c) For $(i-1)t_s \leq t < it_s$ and i odd, x(t)=0. For $it_s \leq t < (i+1)t_s$ and i odd,

$$\Pr(X(t)=1)=p$$
$$\Pr(X(t)=-1)=q=1-p$$

Problem 9.12

(a)

$$\mathbf{P}=\begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.4 & 0.6 & 0 \\ 1 & 0 & 0 \end{bmatrix}=\mathbf{Q}\Lambda\mathbf{Q}^{-1}.$$

Using MATLAB we found,

$$\mathbf{Q}=\begin{bmatrix} -0.4197 & 0.5774 & 0.2792 \\ 0.1570 & 0.5774 & -0.3981 \\ 0.8940 & 0.5774 & 0.8738 \end{bmatrix}, \quad \Lambda=\begin{bmatrix} -0.4695 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.3195 \end{bmatrix}.$$

The limiting form of the k-step transition probability matrix is then found as follows:

$$\lim_{k\to\infty} \mathbf{\Lambda}^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\lim_{k\to\infty} \mathbf{P}^k = \mathbf{Q} \lim_{k\to\infty} \mathbf{\Lambda}^k \mathbf{Q}^{-1} = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.1 \end{bmatrix}.$$

The steady-state distribution is then

$$\pi = \begin{bmatrix} 0.4 & 0.5 & 0.1 \end{bmatrix}.$$

(b) Using MATLAB to calculate $\mathbf{P}^{100}$, we get the same matrix found in part (a):

$$\mathbf{P}^{100} = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.1 \end{bmatrix}.$$

$$\Rightarrow P_{1,3}^{(100)} = 0.1.$$

The interpretation of this result is that for all practical purposes, the process has reached steady state after 100 steps.

(c)

$$\pi(3) = \pi(0)\mathbf{P}^3 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.4 & 0.6 & 0 \\ 1 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0.4274 & 0.4808 & 0.0919 \end{bmatrix}$$

$$\Pr(\text{in state 3 after 3rd step}) = 0.0919.$$

Problem 9.14

(a)

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{12} & 0 & \frac{7}{12} & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{12} & 0 & \frac{7}{12} & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{12} & \frac{1}{6} & \frac{5}{12} & 0 & 0 \\ 0 & 0 & 0 & \frac{7}{12} & 0 & \frac{5}{12} & 0 \\ 0 & 0 & 0 & 0 & \frac{7}{12} & 0 & \frac{5}{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array}$$

(b)

$$\Pr(\text{loses in 3 tosses}) = \Pr(3 \to 2 \to 1 \to 0) = (5/12)^3 = 0.0723.$$

(c)

$$\Pr(\text{loses in 4 tosses}) = \Pr(3 \to 3 \to 2 \to 1 \to 0) = \frac{1}{6}\left(\frac{5}{12}\right)^3 = 0.0121.$$

$$\begin{aligned} \Pr(\text{loses in 5 tosses}) &= \Pr(3 \to 3 \to 3 \to 2 \to 1 \to 0) \\ &+ \Pr(3 \to 4 \to 3 \to 2 \to 1 \to 0) \\ &+ \Pr(3 \to 2 \to 3 \to 2 \to 1 \to 0) \\ &+ \Pr(3 \to 2 \to 1 \to 2 \to 1 \to 0) \\ &= \left(\frac{1}{6}\right)^2 \left(\frac{5}{12}\right)^3 + 3\frac{7}{12}\left(\frac{5}{12}\right)^4 = 0.0548. \end{aligned}$$

$$\Pr(\text{loses in 5 or fewer tosses}) = 0.0723 + 0.0121 + 0.0548 = 0.1392.$$

We can verify this solution using MATLAB by noting that the probability of interest is found by finding the entry in the 4th row and 1st column of $\mathbf{P}^5$.