Solutions to supplementary exercises

Problem 1

For averaged periodograms or Bartlett's method we have:

$$\frac{\Delta f}{f_S} = \frac{0.89}{L} \qquad \text{Frequency Resolution}$$

$$\text{var}\left[\hat{P}_B\left(e^{jw}\right)\right] \approx \frac{1}{K} P_X^2\left(e^{jw}\right) \qquad \text{Variance}$$

$$N = KL$$

Where L is length of samples per a time interval, K is number of time intervals and N is total number of samples. Therefore

$$\nu = \text{Variability=Relative Variance=}\frac{\text{var}\left[\hat{P}_B\left(e^{jw}\right)\right]}{P_X^2\left(e^{jw}\right)} \approx \frac{1}{K} = 0.01 \rightarrow K = 100$$

$$\frac{\Delta f}{f_S} = \frac{0.89}{L} \rightarrow \frac{1}{1000} = \frac{0.89}{L} \rightarrow L = 890$$

$$N = KL = 89000$$

$$t = \frac{N}{f_S} = \frac{89000}{1000} = 89[s]$$

Problem 2

$$\nu = \frac{\operatorname{var}\left[\hat{P}_B\left(e^{jw}\right)\right]}{P_X^2\left(e^{jw}\right)} \approx \frac{1}{K}$$

$$\frac{\Delta f}{f_S} = \frac{0.89}{L} \rightarrow$$

$$\begin{cases} \dfrac{10}{10000} = \dfrac{0.89}{L_{Emilia}} \rightarrow L_{Emilia} = 890 \rightarrow t = \dfrac{N}{f_S} = \dfrac{N}{10000} = 10 \rightarrow N_{Emilia} = 10^5 \\ \dfrac{10}{100000} = \dfrac{0.89}{L_{Emil}} \rightarrow L_{Emil} = 8900 \rightarrow t = \dfrac{N}{f_S} = \dfrac{N}{100000} = 10 \rightarrow N_{Emil} = 10^6 \end{cases}$$

$$N = KL \rightarrow \begin{cases} K_{Emilia} = \dfrac{10^5}{890} = \dfrac{10^4}{89} \rightarrow \nu_{Emilia} = 0.0089 \\ K_{Emilia} = \dfrac{10^6}{8900} = \dfrac{10^4}{89} \rightarrow \nu_{Emil} = 0.0089 \end{cases}$$

As we can see there is no difference between normalized variances.

Problem 3

It is easy to recognize the underlying method is Blackman-Tukey method with following properties:

$$\hat{P}_{BT}\left(e^{jw}\right)=\sum_{-M}^{M}\hat{r}_X(k)w(k)e^{-jw}$$

$$\text{var}\left[\hat{P}_{BT}\left(e^{jw}\right)\right]\approx P_X\left(e^{jw}\right)\frac{1}{N}\sum_{-M}^{M}w^2(k)\rightarrow \text{v}\approx\frac{1}{N}\sum_{-M}^{M}w^2(k)$$

Therefore, we will have

$$N=10000$$

$$M=100$$

$$w(n)=\begin{cases}e^{-0.1|n|} & ; \quad |n|\le 100\\ 0 & ; \quad |n|>100\end{cases}$$

$$\text{v}\approx\frac{1}{N}\sum_{-M}^{M}w^2(k)=\frac{1}{10^4}\sum_{k=-100}^{100}e^{-0.2|k|}=\frac{1}{10^4}\left(2\sum_{k=0}^{100}e^{-0.2k}-1\right)$$

$$=\frac{1}{10^4}\left(2\left(\frac{e^{-20.2}-1}{e^{-0.2}-1}\right)-1\right)$$

$$=10.0333*10^{-4}$$

Problem 4

a)

$$x\left[n\right]+0.6x\left[n-1\right]-0.2x\left[n-2\right]=e\left[n\right]$$
$$x\left[n\right]=-0.6x\left[n-1\right]+0.2x\left[n-2\right]+e\left[n\right]$$
$$R[k]=E\left[x\left[n-k\right]x\left[n\right]\right]=E\left[x\left[n-k\right]\left(-0.6x\left[n-1\right]+0.2x\left[n-2\right]+e\left[n\right]\right)\right]$$
$$=-0.6R[k-1]+0.2R[k-2]+E\left[x\left[n-k\right]e\left[n\right]\right]$$
$$k=0: E\left[x\left[n-k\right]e\left[n\right]\right]=E\left[\left(-0.6x\left[n-1\right]+0.2x\left[n-2\right]+e\left[n\right]\right)e\left[n\right]\right]=1$$
$$k>0: E\left[x\left[n-k\right]e\left[n\right]\right]=0$$

$$k=0: R[0]=-0.6R[1]+0.2R[2]+1 \rightarrow R[0]+0.6R[1]-0.2R[2]=1$$
$$k=1: R[1]=-0.6R[0]+0.2R[1] \rightarrow 0.6R[0]-0.8R[1]=0$$
$$k=2: R[2]=-0.6R[1]+0.2R[0] \rightarrow -0.2R[0]+0.6R[1]+R[2]=0$$
$$|k|>2: R[k]=-0.6R[k-1]+0.2R[k-2]$$

$$\begin{bmatrix} 1 & 0.6 & -0.2 \\ 0.6 & -0.8 & 0 \\ 0.2 & 0.6 & 1 \end{bmatrix}\begin{bmatrix} R[0] \\ R[1] \\ R[2] \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} R[0] \\ R[1] \\ R[2] \end{bmatrix}=\begin{bmatrix} 2.4 \\ -1.8 \\ 1.6 \end{bmatrix}$$

Attention Please!

$$H[z] = \frac{1}{1+0.6z^{-1}-0.2z^{-2}}$$

$$P_X[z] = H[z]H^*[\frac{1}{z^*}]P_E[z]$$

$$= \left(\frac{1}{1+0.6z^{-1}-0.2z^{-2}}\right)\left(\frac{1}{1+0.6z-0.2z^{2}}\right)1$$

$$= \frac{1}{1.4-0.6(z+z^{-1})-0.2(z^2+z^{-2})}$$

$$P_X(e^{j\omega}) = \frac{1}{1.4-1.2\cos(\omega)-0.4\cos(2\omega)}$$

b)

$$x[n] = e[n] + 0.8e[n-1] + 0.2e[n-2]$$

$$R[k] = E\left[x[n-k]x[n]\right] = E\left[\left(e[n-k]+0.8e[n-k-1]+0.2e[n-k-2]\right)\left(e[n]+0.8e[n-1]+0.2e[n-2]\right)\right]$$

$$= 0.2R_{EE}(k+2) + 0.96R_{EE}(k+1) + 1.68R_{EE}(k) + 0.96R_{EE}(k-1) + 0.2R_{EE}(k-2)$$

$k = 0: R[0] = 1.68R_{EE}[0] = 1.68$

$k = 1: R[1] = R[-1] = 0.96R_{EE}[0] = 0.96$

$k = 2: R[2] = R[-2] = 0.4R_{EE}[0] = 0.2$

$|k| > 2: R[k] = 0$

$$P_X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} R_{XX}[n]e^{-j\omega n} = \sum_{n=-2}^{2} R_{XX}[n]e^{-j\omega n} = 1.68 + 0.96\left(e^{j\omega}+e^{-j\omega}\right) + 0.2\left(e^{2j\omega}+e^{-2j\omega}\right)$$

$$= 1.68 + 1.92\cos(\omega) + 0.2\cos(2\omega)$$

or we could simply write

$$x[n] = e[n] + 0.8e[n-1] + 0.2e[n-2]$$

$$\rightarrow X[z] = E[z](1+0.8z^{-1}+0.2z^{-2})$$

$$\rightarrow H[z] = 1+0.8z^{-1}+0.2z^{-2}$$

$$P_X[z] = H[z]H^*[\frac{1}{z^*}]P_E[z]$$

$$= \left(1+0.8z^{-1}+0.2z^{-2}\right)\left(1+0.8z^{1}+0.2z^{2}\right)1$$

$$= 1.68 + 0.96(z+z^{-1}) + 0.2(z^2+z^{-2})$$

$$P_X(e^{j\omega}) = 1.68 + 1.92\cos(\omega) + 0.4\cos(2\omega)$$

Problem 6

$$x[n] + a_1 x[n-1] + a_2 x[n-2] = e[n]$$

Multiplying by $x[n-k];\ k \geq 0$ and taking the expectation leads to the relation

$$r[k] + a_1 r[k-1] + a_2 r[n-2] = E[x[n-k]e[n]]$$

$$\begin{matrix} k=0: \\ k=1: \\ k=2: \\ k=3: \\ k=4: \end{matrix} \begin{bmatrix} r[0] & r[1] & r[2] \\ r[1] & r[0] & r[1] \\ r[2] & r[1] & r[0] \\ r[3] & r[2] & r[1] \\ r[4] & r[3] & r[2] \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sigma_e^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 7.73 & 6.8 & 4.75 \\ 6.8 & 7.73 & 6.8 \\ 4.75 & 6.8 & 7.73 \\ 2.36 & 4.75 & 6.8 \\ 0.23 & 2.36 & 4.75 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} \sigma_e^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{b}}$$

The above equation which has the form of Ax=b, can be solved by

$$\mathbf{x} = \left(\mathbf{A}^T \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{b}$$

$$\rightarrow \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \sigma_e^2 \begin{bmatrix} 1.1061 \\ -1.6451 \\ 0.7655 \end{bmatrix} \rightarrow \begin{cases} \sigma_e^2 \approx 0.9041 \\ a_1 \approx -1.4873 \\ a_2 \approx 0.6921 \end{cases}$$

Problem 8

$$d[n]-0.8d[n-1]=e[n]$$

Multiplying by $d[n-k];\ k\geq 0$ and taking the expectation leads to the relation

$$r[k]-0.8r[k-1]=E[d[n-k]e[n]]$$

$$\begin{matrix} k=0: \\ k=1: \end{matrix} \begin{bmatrix} 1 & -0.8 \\ -0.8 & 1 \end{bmatrix}\begin{bmatrix} r[0] \\ r[1] \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\rightarrow \begin{cases} r[0]=2.7778 \\ r[k]=0.8r[k-1];\ k\geq 1 \end{cases}$$

Also we have

$$D[z](1+0.5z^{-1})+V[z]=X[z]\rightarrow d[n]+0.5d[n-1]+v[n]=x[n]$$

If we estimate $\hat{d}[n]=w_0x[n]+w_1x[n-1]$, then the squared error can be estimated as follows

$$\begin{aligned} E[e^2] &= E[(d[n]-\hat{d}[n])^2] \\ &= E\left[\left(d[n]-w_0\left(d[n]+0.5d[n-1]+v[n]\right)+w_1\left(d[n-1]+0.5d[n-2]+v[n-1]\right)\right)^2\right] \\ &= w_0^2\left(1.25r[0]+r[1]+0.1\right)+w_1^2\left(1.25r[0]+r[1]+0.1\right)+w_0\left((w_1-2)r[0]+(2.5w_1-1)r[1]+w_1r[2]\right) \\ &+w_1\left(-2r[1]-r[2]\right)+r[0] \\ &= 5.79445w_0^2+5.79445w_1^2+10.1111w_0w_1-7.7778w_0-6.2222w_1+2.7778 \end{aligned}$$

Taking the derivative of $E[e^2]$ according to w_0 and w_1, we get

$$\frac{\partial E[e^2]}{\partial w_0} = 11.5889 w_0 + 10.1111 w_1 - 7.7778 = 0$$

$$\frac{\partial E[e^2]}{\partial w_1} = 11.5889 w_1 + 10.1111 w_0 - 6.2222 = 0$$

$$\rightarrow \begin{bmatrix} 11.5889 & 10.1111 \\ 10.1111 & 11.5889 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 7.7778 \\ 6.2222 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0.8489 \\ -0.2037 \end{bmatrix}$$