Solutions to Chapter 11 Home Exercises

Problem 11.1

(a) The impulse response of the integrator is $h(t) = \text{rect}(\frac{t-t_o/2}{t_o})$. Hence the transfer function is

$$H(f) = t_o \text{sinc}(ft_o)e^{-j2\pi f t_o}.$$

The PSD of the output is

$$S_{Y,Y}(f) = S_{X,X}(f)|H(f)|^2 = \frac{N_o t_o^2}{2}\text{sinc}^2(ft_o).$$

(b) Taking the inverse transforms of the above result

$$\begin{aligned} R_{Y,Y}(\tau) &= F^{-1}[S_{Y,Y}(f)] \\ &= \frac{N_o t_o}{2} F^{-1}[t_o \text{sinc}^2(ft_o)] \\ &= \frac{N_o t_o}{2} \text{tri}(t/t_o). \end{aligned}$$

The total power in the output is:

$$P_Y = R_{Y,Y}(0) = \frac{N_o t_o}{2}.$$

(c)The noise equivalent bandwidth, B_{neq}, will satisfy

$$\begin{aligned} \frac{N_o t_o^2}{2} \cdot 2B_{neq} &= P_Y = \frac{N_o t_o}{2} \\ \Rightarrow B_{neq} &= \frac{1}{2t_o}. \end{aligned}$$

Problem 11.14

The given function is not a valid autocorrelation since $|R_{X,X}[k]|$ is not less than $R_{X,X}[0]$ for all k.

Problem 11.7

From example 10.5, the PSD of the random telegraph process is

$$S_{X,X}(f) = \frac{A}{4}\delta(f) + \frac{A}{4} \cdot \frac{c}{c^2 + (\pi f)^2}.$$

The output PSD is given by

$$S_{Y,Y}(f) = S_{X,X}(f)|H(f)|^2.$$

For the filter given,

$$h(t) = be^{-at}u(t) \leftrightarrow H(f) = \frac{b}{a + j2\pi f}.$$

Therefore, the output PSD is

$$\begin{aligned} S_{Y,Y}(f) &= \left(\frac{A}{4}\delta(f) + \frac{A}{4} \cdot \frac{c}{c^2 + (\pi f)^2}\right)\left(\frac{b^2}{a^2 + (2\pi f)^2}\right) \\ &= \frac{Ab^2}{4a^2}\delta(f) + \frac{Acb^2}{4} \cdot \frac{1}{(c^2 + (\pi f)^2)(a^2 + (2\pi f)^2)}. \end{aligned}$$

Problem 11.9

$$S_{Y,Y}(f) = S_{X,X}(f)|H(f)|^2 = \frac{a}{1 + (f/f_o)^2} \cdot b^2 \left[\text{rect}\left(\frac{f - f_c}{f_2 - f_1}\right) + \text{rect}\left(\frac{f + f_c}{f_2 - f_1}\right)\right],$$

where $f_c = (f_1 + f_2)/2$. Assuming $f_2 - f_1 << f_o$, the input PSD will be approximately constant over the passband of the filter. In that case, the output PSD will be well approximated by

$$\begin{aligned} S_{Y,Y}(f) &\approx S_{X,X}(f_c)|H(f)|^2 = \frac{ab^2}{1 + (f_c/f_o)^2} \cdot \left[\text{rect}\left(\frac{f - f_c}{f_2 - f_1}\right) + \text{rect}\left(\frac{f + f_c}{f_2 - f_1}\right)\right] \\ \Rightarrow &= \frac{2ab^2(f_2 - f_1)}{1 + (f_c/f_o)^2}\text{sinc}((f_2 - f_1)\tau)\cos(\omega_c \tau). \end{aligned}$$

Problem 11.12

(a)

$$\begin{aligned}
R_{Y_1,Y_2}(\tau) &= E[Y_1(t)Y_2(t+\tau)] \\
&= E[\int h_1(u)N(t-u)du \int h_2(v)N(t+\tau-v)dv] \\
&= \int\int h_1(u)h_2(v)E[N(t-u)N(t+\tau-v)]dudv \\
&= \int\int h_1(u)h_2(v)R_{N,N}(\tau-v+u)dudv \\
&= \int\int h_1(u)h_2(v)\delta(\tau-v+u)dudv \\
&= \int h_1(u)h_2(u+\tau)du \\
&= h_1(-\tau) * h_2(\tau).
\end{aligned}$$

(b)

$$S_{Y_1,Y_2}(f) = F[h_1(-\tau) * h_2(\tau)] = H_1^*(f)H_2(f).$$

(c) The two processes are independent at the same sampling times if $R_{Y_1,Y_2}(0) = 0$. From the results of part (a), this constraint is expressed in the time domain as

$$R_{Y_1,Y_2}(0) = 0 \Rightarrow \int h_1(t)h_2(t)dt = 0.$$

In words, the impulse responses of the filters must be orthogonal. Transforming this to the frequency domain, the constraint becomes

$$\int S_{Y_1,Y_2}(f)df = 0 \Rightarrow \int H_1^*(f)H_2(f)df = 0.$$

Hence the transfer functions must be orthogonal.

(d) The two processes are independent at arbitrary sampling times if $R_{Y_1,Y_2}(\tau) = 0$ for all τ. This leads to

$$\int h_1(t)h_2(t+\tau)dt = 0.$$

In the frequency domain, we must have $S_{Y_1,Y_2}(f) = 0$ for all f. This leads to

$$H_1^*(f)H_2(f) = 0.$$

In this case, the transfer functions must be non-overlapping in frequency.

Problem 11.22

$$H(f) = \frac{4}{10 + j2\pi f} \leftrightarrow h(t) = 4e^{-10t}u(t).$$

The noise equivalent BW is found as follows:

$$\begin{aligned}
\int_0^\infty |H(f)|^2 df &= \frac{1}{2}\int_{-\infty}^\infty |H(f)|^2 df = \frac{1}{2}\int_{-\infty}^\infty h^2(t)dt \\
&= 8\int_0^\infty e^{-20t}dt = \frac{2}{5}. \\
|H(0)|^2 &= \frac{4}{25} \\
\Rightarrow B_{neq} &= \frac{1}{|H(0|^2}\int_0^\infty |H(f)|^2 df = \frac{5}{2}.
\end{aligned}$$

The 3dB BW occurs when

$$|H(f_3)|^2 = \frac{1}{2}|H(0)|^2 \Rightarrow f_3 = \frac{5}{\pi}.$$

The ratio is then

$$\frac{B_{neq}}{f_3} = \frac{5/2}{5/\pi} = \frac{\pi}{2}.$$