Solutions to Chapter 3 Home Exercises

Problem 3.14

a) Since

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$$

$$c\int_{-\infty}^{+\infty} \frac{1}{x^2+4}\,dx = 1$$

$$\frac{c}{4}\arctan\left(\frac{x}{2}\right)\Big|_{-\infty}^{+\infty} = 1 \rightarrow \frac{c}{4}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = 1 \rightarrow c = \frac{4}{\pi}, f_X(x) = \frac{4}{\pi\left(x^2+4\right)}$$

b)

$$\Pr(X>2) = \int_{2}^{+\infty} \frac{4}{\pi\left(x^2+4\right)}\,dx = \frac{1}{\pi}\arctan\left(\frac{x}{2}\right)\Big|_{2}^{+\infty} = \frac{1}{\pi}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = 0.25$$

c)

$$\Pr(X<3) = \int_{-\infty}^{3} \frac{4}{\pi\left(x^2+4\right)}\,dx = \frac{1}{\pi}\arctan\left(\frac{x}{2}\right)\Big|_{-\infty}^{3} = \frac{1}{\pi}\left(\arctan\frac{3}{2}+\frac{\pi}{2}\right) = 0.8128$$

d)

$$\Pr(X<3 \mid X>2) = \frac{\Pr(2<X<3)}{\Pr(X>2)}$$

$$= \frac{\int_{2}^{3} \frac{4}{\pi\left(x^2+4\right)}\,dx}{0.25}$$

$$= \frac{\frac{1}{\pi}\arctan\left(\frac{x}{2}\right)\Big|_{2}^{3}}{0.25}$$

$$= \frac{\frac{1}{\pi}\left(\arctan\frac{3}{2}-\frac{\pi}{4}\right)}{0.25} = 0.2513$$

Problem 3.20

Since

$$\exp(-(ax^2+bx+c)) = \exp\left(-\left(\frac{(x+\frac{b}{2a})^2+\frac{c}{a}-\frac{b^2}{4a^2}}{\frac{1}{a}}\right)\right) = \exp(\frac{b^2}{4a}-c)\exp\left(-\frac{(x+\frac{b}{2a})^2}{\frac{1}{a}}\right) ,$$

then we have

$$\begin{aligned}\int_{-\infty}^{\infty}\exp(-(ax^2+bx+c))dx &= \int_{-\infty}^{\infty}\exp(\frac{b^2}{4a}-c)*\exp\left(-\frac{(x+\frac{b}{2a})^2}{\frac{1}{a}}\right)dx \\ &= \exp(\frac{b^2}{4a}-c)\sqrt{\frac{\pi}{a}}*\frac{1}{\sqrt{\frac{\pi}{a}}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x+\frac{b}{2a})^2}{\frac{1}{a}}dx\right) .\end{aligned}$$

Since

$$\frac{1}{\sqrt{\frac{\pi}{a}}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x+\frac{b}{2a})^2}{\frac{1}{a}}\right)dx = 1,$$

then

$$\int_{-\infty}^{\infty}\exp(-(ax^2+bx+c))dx = \sqrt{\frac{\pi}{a}}\exp(\frac{b^2}{4a}-c) .$$

Problem 3.35

(a)

$$f_{X|X>0}(x) = \begin{cases} \frac{f_X(x)}{P_r(X>0)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{f_X(x)}{Q(0)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 2f_X(x) & x > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{2}{\sqrt{2\pi\sigma^2}} \exp(-\frac{x^2}{2\sigma^2}) & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$f_{X||X|<3}(x) = \begin{cases} \frac{f_X(x)}{P_r(-3<X<3)} & -3 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{f_X(x)}{1-2Q(\frac{3}{\sigma})} & -3 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{1}{1-2Q(\frac{3}{\sigma})} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{x^2}{2\sigma^2}) & -3 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$f_{X||X|>3}(x) = \begin{cases} \frac{f_X(x)}{P_r(|X|>3)} & |X| > 3 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{f_X(x)}{2Q(\frac{3}{\sigma})} & |X| > 3 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{1}{2Q(\frac{3}{\sigma})} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{x^2}{2\sigma^2}) & |X| > 3 \\ 0 & \text{otherwise} \end{cases}$$

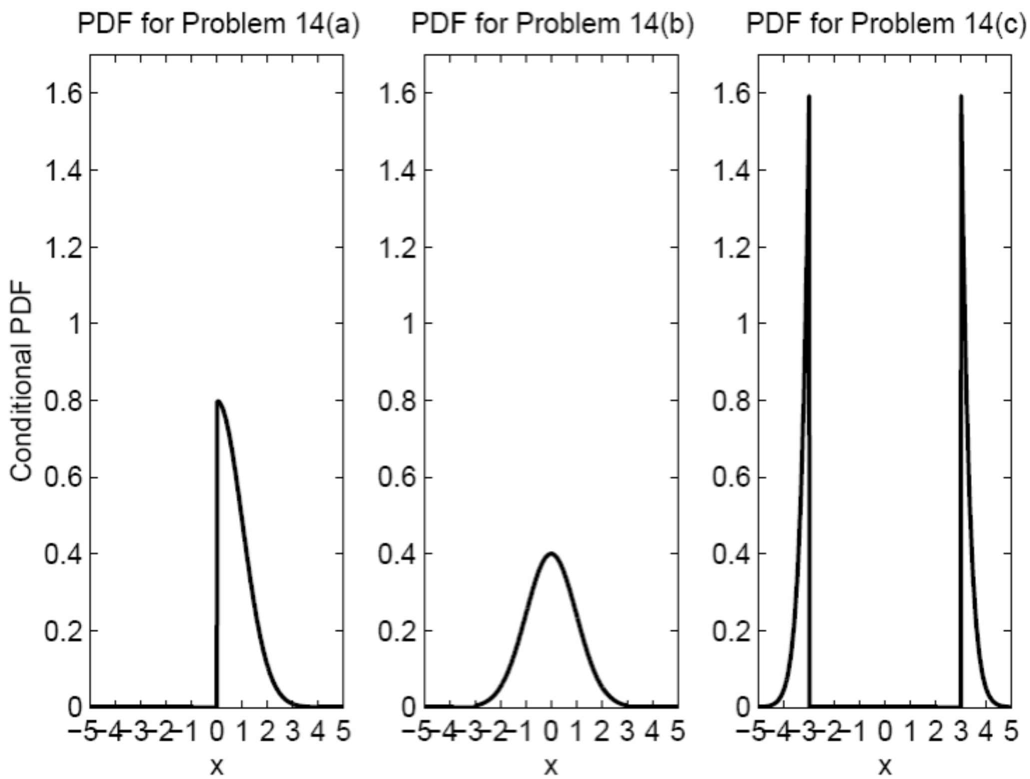

Figure 1 PDF plots for problem 3.14; (σ^2=1).

Problem 3.46

```
function Es_Pr=PrEs(x1,x2,sigma,m)

%Es_Pr=PrEs(x1,x2,sigma,m)
%This program will calculate the probability Pr(x1=<X=<x2)
%for X which is Gaussian random variable

%%%%    Inputs:
%%%%    x1:     lower bound
%%%%    x2:     upper bound
%%%%    sigma:  standard deviation
%%%%    m:      mean value

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Es_Pr=0.5*erfc(((x1-m)/sigma)/sqrt(2))-0.5*erfc(((x2-m)/sigma)/sqrt(2));
```