Solutions to Chapter 5 Home Exercises

Problem 5.10

$$f_{X,Y}(x,y) = \frac{1}{8\pi}\exp\left(-\frac{(x-1)^2+(y+1)^2}{8}\right)$$

(a)$Pr(X>2, Y<0)$

$$\begin{aligned}
Pr(X>2,Y<0) &= \int_2^\infty\int_{-\infty}^0 \frac{1}{8\pi}\exp\left(-\frac{(x-1)^2+(y+1)^2}{8}\right)\\
&= \int_2^\infty \frac{1}{\sqrt{8\pi}}\exp(-\frac{(x-1)^2}{8})\,dx\int_{-\infty}^0 \frac{1}{\sqrt{8\pi}}\exp(-\frac{(y+1)^2}{8})\,dy\\
&= Q\left(\frac{2-1}{2}\right)\Phi\left(\frac{0+1}{2}\right) = Q\left(\frac{1}{2}\right)\Phi\left(\frac{1}{2}\right)\\
&= Q(1/2)\,(1-Q(1/2))
\end{aligned}$$

(b)$Pr(0<X<2, |Y+1|>2)$

$$\begin{aligned}
&= Pr(\{0<X<2,Y>1\}\cup\{0<X<2,Y<-3\})\\
&= \int_0^2\int_{-\infty}^{-3} f_{X,Y}(x,y)dx\,dy + \int_0^2\int_1^\infty f_{X,Y}(x,y)dx\,dy\\
&= \int_0^2\frac{1}{\sqrt{8\pi}}\,exp(-\frac{(x-1)^2}{8})\,dx\left(\int_{-\infty}^{-3}\frac{1}{\sqrt{8\pi}}\exp(-\frac{(y+1)^2}{8})\,dy + \int_1^\infty\frac{1}{\sqrt{8\pi}}\exp(-\frac{(y+1)^2}{8})\,dy\right)\\
&= \left(Q\left(\frac{0-1}{2}\right)-Q\left(\frac{2-1}{2}\right)\right)\left(\Phi\left(\frac{-3+1}{2}\right)+Q\left(\frac{1+1}{2}\right)\right)\\
&= \left(Q\left(-\frac{1}{2}\right)-Q\left(\frac{1}{2}\right)\right)(\Phi(-1)+Q(1))\\
&= 2Q(1)\left(1-2Q\left(\frac{1}{2}\right)\right)
\end{aligned}$$

(c) $Pr(Y > X)$

$$Pr(Y > X) = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f_{X,Y}(x,y)dx\,dy$$

This integral is easier to do with a change of variables. Let us use the substitution

$$\begin{aligned}
u &= x - y \\
v &= x + y \\
J\begin{pmatrix} u & v \\ x & y \end{pmatrix} &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\
\left| det \begin{bmatrix} u & v \\ x & y \end{bmatrix} \right| &= 2 \\
f_{U,V}(u,v) &= \frac{f_{X,Y}(x,y)}{2} \\
Pr(Y > X) &= Pr(U < 0) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{0} \frac{1}{8\pi}\exp\left(-\frac{(x-1)^2+(y+1)^2}{8}\right)\frac{1}{2}\,du\,dv \\
&= \frac{1}{16\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \exp\left(-\frac{(u-v-2)^2+(u+v+2)^2}{32}\right)du\,dv \\
&= \frac{1}{16\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \exp\left(-\frac{(u^2+v^2+4-4u}{16}\right)du\,dv \\
\\
&= \frac{1}{16\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \exp\left(-\frac{(u-2)^2+v^2}{16}\right)du\,dv \\
&= \frac{1}{16\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{v^2}{16}\right)dv \int_{-\infty}^{0} \exp\left(-\frac{(u-2)^2}{16}\right)du \\
&= \frac{1}{16\pi}\sqrt{16\pi}\sqrt{16\pi}\,\Phi\left(\frac{0-2}{\sqrt{8}}\right) \\
&= Q(\frac{1}{\sqrt{2}})
\end{aligned}$$

Problem 5.30

Note that any linear transformation of jointly Gaussian random variables produces jointly Gaussian random variables. So U, V are both Gaussian random variables. And if $\rho_{U,V} = 0$, U and V are independent.

$$E[U] = aE[X] + bE[Y] = 0 \ .$$

$$E[V] = cE[X] + dE[Y] = 0 \ .$$

$$E[UV] = acE[X^2] + bdE[Y^2] + (ad + bc)E[XY] = ac + bd + \rho(ad + bc) \ .$$

Thus, $\rho_{U,V} = 0$ if

$$ac + bd + \rho(ad + bc) = 0 \ .$$

So, U and V are independent, if

$$ac + bd = -\rho(ad + bc) \ .$$

Problem 5.60

$$\begin{aligned}
Z &= X^2 + Y^2 \\
W &= XY \\
X &= \frac{\sqrt{Z+2W} + \sqrt{Z-2W}}{2} \\
Y &= \frac{\sqrt{Z+2W} - \sqrt{Z-2W}}{2} \\
f_X(x) &= \frac{1}{\pi(1+x^2)} \\
f_Y(y) &= \frac{1}{\pi(1+y^2)} \\
J\begin{pmatrix} z & w \\ x & y \end{pmatrix} &= \begin{pmatrix} 2x & 2y \\ y & x \end{pmatrix} \\
\left| det\left[J\begin{pmatrix} z & w \\ x & y \end{pmatrix} \right] \right| &= 2|x^2 - y^2| \\
f_{Z,W}(z,w) &= \frac{f_X(x) f_Y(y)}{2|x^2-y^2|} \\
&= \frac{1}{2\pi^2} \frac{1}{(1+x^2)(1+y^2)|x^2-y^2|} \\
&= \frac{1}{2\pi^2} \frac{1}{(1+x^2+y^2+x^2y^2)|x^2-y^2|}
\end{aligned}$$

$$= \frac{1}{2\pi^2} \frac{1}{(1+z^2+w^2)\sqrt{z^2-4w^2}}$$