Solutions to Chapter 8 Home Exercises

Problem 8.12

Define $X(t)$ and $Y(t)$ according to:

$$\begin{aligned} X(t) &= A(t)\cos(t), \\ Y(t) &= B(t)\sin(t), \\ \mu_A(t) &= \mu_B(t) = 0, \\ R_{A,A}(\tau) &= R_{B,B}(\tau) = R(\tau), \\ R_{A,B}(\tau) &= 0. \end{aligned}$$

Then,

$$\begin{aligned} E[X(t)] &= 0. \\ E[Y(t)] &= 0. \\ R_{X,X}(t_1,t_2) &= E[A(t_1)A(t_2)\cos(t_1)\cos(t_2)] \\ &= R(t_2-t_1)\cos(t_1)\cos(t_2) \\ R_{Y,Y}(t_1,t_2) &= E[B(t_1)B(t_2)\sin(t_1)\sin(t_2)] \\ &= R(t_2-t_1)\sin(t_1)\sin(t_2) \\ R_{Z,Z}(t_1,t_2) &= E[(X(t_1)+Y(t_1))(X(t_2)+Y(t_2))] \\ &= R_{X,X}(t_1,t_2)+R_{Y,Y}(t_1,t_2)+R_{X,Y}(t_1,t_2)+R_{Y,X}(t_1,t_2) \\ &= R(t_2-t_1)[\cos(t_1)\cos(t_2)+\sin(t_1)\sin(t_2)]\ (\text{since} R_{X,Y}(\tau)=R_{Y,X}(\tau)=0) \\ &= R(t_2-t_1)\cos(t_2-t_1). \end{aligned}$$

Therefore, for this example, $Z(t) = X(t)+Y(t)$ is WSS, while $X(t)$ and $Y(t)$ are not WSS.

Problem 8.13

$$\begin{aligned}
X(t) &= A(t)\cos(\omega_0 t + \Theta) \\
E[X(t)] &= E[A(t)]E[\cos(\omega_0 t + \Theta)] = 0 \\
R_{X,X}(t_1,t_2) &= E[A(t_1)A(t_2)]E[\cos(\omega_0 t_1 + \Theta)\cos(\omega_0 t_2 + \Theta)] \\
&= \frac{1}{2}R_{A,A}(t_2 - t_1)\{E[\cos(\omega_0(t_2 - t_1))] + E[\cos(\omega_0(t_1 + t_2) + 2\Theta)]\} \\
&= \frac{1}{2}R_{A,A}(t_2 - t_1)\cos(\omega_0(t_2 - t_1)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
E[Y(t)] &= 0 \\
R_{Y,Y}(t_1,t_2) &= \frac{1}{2}R_{A,A}(t_2 - t_1)\cos((\omega_0 + \omega_1)(t_2 - t_1)).
\end{aligned}$$

Hence, both $X(t)$ and $Y(t)$ are WSS. If $Z(t) = X(t) + Y(t)$,

$$\begin{aligned}
E[z(t)] &= E[X(t)] + E[Y(t)] = 0, \\
R_{Z,Z}(t_1,t_2) &= R_{X,X}(t_1,t_2) + R_{Y,Y}(t_1,t_2) + R_{X,Y}(t_1,t_2) + R_{Y,X}(t_1,t_2), \\
R_{X,Y}(t_1,t_2) &= E[A(t_1)A(t_2)]E[\cos(\omega_0 t_1 + \Theta)\cos((\omega_0 + \omega_1)t_2 + \Theta)], \\
&= \frac{1}{2}R_{A,A}(t_2 - t_1)\cos(\omega_0(t_2 - t_1) + \omega_1 t_2), \\
R_{Y,X}(t_1,t_2) &= R_{A,A}(t_2 - t_1)\cos(\omega_0(t_1 - t_2) + \omega_1 t_1).
\end{aligned}$$

Since $R_{Z,Z}(t_1,t_2)$ is not a function of $t_1 - t_2$, $Z(t)$ is not WSS.

Problem 8.16

(a) Since T is uniformly distributed over one period of $s(t)$, for any time instant t, $X(t) = s(t-T)$ will be equally likely to take on any of the values in one period of $s(t)$. Given the linear functional form of $s(t)$, $X(t)$ will be uniform over $(-1, 1)$.

$$f_X(x;t) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) $E[X(t)] = 0$ since the PDF above is symmetric about zero.
(c)

$$\begin{aligned} R_{X,X}(t_1, t_2) &= E[s(t_1 - T)s(t_2 - T)] \\ &= \int_0^1 s(t_1 - u)s(t_2 - u)du \\ &= \int_0^1 s(v)s(v + t_2 - t_1)dv \\ &= s(t) * s(-t)\Big|_{t=t_2-t_1}. \end{aligned}$$

This is the time correlation of a triangle wave with itself which will result in the periodic signal shown in Figure 2.

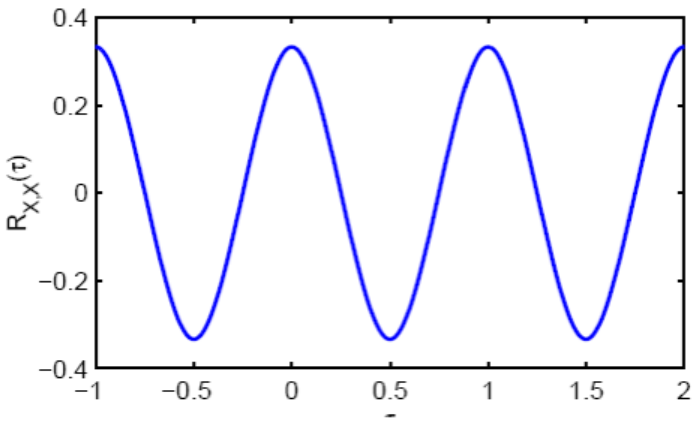

Figure 2: Autocorrelation function for process of Exercise 8.12

(d) The process is WSS.

Problem 8.20

Since

$$\lim_{k\to\infty} R_{X,X}[k] = \mu_X = 0,$$

the process is ergodic in the mean.

Problem 8.35

Using probability generating functions:

$$\begin{aligned}
H_X(z) &= E[z^{X(t)}] = E[z^{\sum_{i=1}^n X_i(t)}] = E\left[\prod_{i=1}^n z^{X_i(t)}\right] = \prod_{i=1}^n E[z^{X_i(t)}] = \prod_{i=1}^n H_{X_i}(z). \\
\Pr(X_i(t) = k) &= \frac{(\lambda_i t)^k}{k!} e^{-\lambda_i t}. \\
H_{X_i}(z) &= \sum_{k=0}^{\infty} \frac{(\lambda_i z t)^k}{k!} e^{-\lambda_i t} = e^{\lambda_i z t} e^{-\lambda_i t} = e^{\lambda_i t(z-1)}. \\
H_X(z) &= \prod_{i=1}^n e^{\lambda_i t(z-1)} = \exp\left(\left(\sum_{i=1}^n \lambda_i\right) t(z-1)\right).
\end{aligned}$$

Define $\lambda = \sum_{i=1}^n \lambda_i$. Then $H_X(z) = \exp(\lambda t(z-1))$ which is the probability generating function of a Poisson random variable. Therefore $X(t)$ is a Poisson proces with arrival rate $\lambda = \sum_{i=1}^n \lambda_i$.

Problem 8.41

Expected number of strikes is st.
(a) In one minute, $st = \frac{1}{3} \cdot 1 = \frac{1}{3}$.
(b) In ten minute, $st = \frac{1}{3} \cdot 10 = \frac{10}{3}$.
(c) Average time between strikes is $\frac{1}{s} = 3$ minutes.