Solutions to Chapter 9 Home Exercises

Problem 9.9

(a)

$$\begin{aligned}
\Pr(X_2 = 1, X_1 = 0) &= \Pr(X_2 = 1 | X_1 = 0) \Pr(X_1 = 0) = p \cdot \Pr(X_1 = 0). \\
\Pr(X_1 = 0) &= \Pr(X_1 = 0 | X_0 = 0) \Pr(X_0 = 0) + \Pr(X_1 = 0 | X_0 = 1) \Pr(X_0 = 1) \\
&= (1-p)s + q(1-s). \\
\Rightarrow \Pr(X_2 = 1, X_1 = 0) &= p(1-p)s + pq(1-s).
\end{aligned}$$

(b)

$$\begin{aligned}
\Pr(X_1 = 1 | X_0 = 0, X_2 = 0) &= \frac{\Pr(X_2 = 0 | X_1 = 1, X_0 = 0) \Pr(X_1 = 1 | X_0 = 0)}{\Pr(X_2 = 0 | X_0 = 0)} \\
&= \frac{P_{1,0}^{(1)} P_{0,1}^{(1)}}{P_{0,0}^{(2)}} \\
&= \frac{qp}{(1-p)^2 + pq}.
\end{aligned}$$

(c)

$$\begin{aligned}
\Pr(X_2 = X_1) &= \Pr(X_2 = 0, X_1 = 0) + \Pr(X_2 = 1, X_1 = 1) \\
&= (1-p) \Pr(X_1 = 0) + (1-q) \Pr(X_1 = 1) \\
&= (1-p)[(1-p)s + q(1-s)] + (1-q)[ps + (1-q)(1-s)] \\
&= (1-p)^2 s + q(1-p)(1-s) + p(1-q)s + (1-q)^2(1-s). \\
\Pr(X_1 = X_0) &= \Pr(X_0 = 0, X_1 = 0) + \Pr(X_0 = 1, X_1 = 1) \\
&= (1-p)s + (1-q)(1-s).
\end{aligned}$$

These expressions are not the same, so $\Pr(X_1 = X_0) \neq \Pr(X_2 = X_1)$.

Problem 9.23

(a)

$$\mathbf{P} = \begin{array}{c} \\ \text{go} \\ \text{skip} \end{array} \begin{array}{c} \begin{array}{cc} \text{go} & \text{skip} \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} \end{array}$$

(b)

$$\Pr(\text{go Friday}|\text{went Wednesday}) = \frac{1}{2}.$$

(c)

$$\mathbf{P}^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{9}{16} & \frac{7}{16} \end{bmatrix}$$

$$\Pr(\text{go Friday}|\text{went Monday}) = P_{0,0}^{(2)} = \frac{5}{8}.$$

(d) $\mathbf{P} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1}$ where

$$\mathbf{Q} = \frac{1}{4}\begin{bmatrix} 4 & 2 \\ 4 & -3 \end{bmatrix}, \boldsymbol{\Lambda} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{Q}^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 4 & -4 \end{bmatrix}.$$

The limiting form of the eigenvalue matrix is

$$\lim_{k\to\infty} \boldsymbol{\Lambda}^k = \lim_{k\to\infty} \begin{bmatrix} 1 & 0 \\ 0 & \left(-\frac{1}{4}\right)^k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, the limiting form of the k-step transition matrix is

$$\lim_{k\to\infty} \mathbf{P}^k = \frac{1}{20}\begin{bmatrix} 4 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 4 & -4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}.$$

Since both rows are equal, a steady state distribution does exist and it is given by

$$\pi = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \end{bmatrix}.$$

Problem 9.25

(a) The first three states communicate with period 1.
(b) All states communicate with period = 3.
(c) All states communicate with period = 2.

Problem 9.19

(a) Let N = number of transmissions.

$$\Pr(N = n) = q^{n-1}(1-q) n = 1, 2, 3, \ldots$$

$$E[N] = \sum_{n=1}^{\infty} n q^{n-1}(1-q) = \frac{1}{1-q}.$$

(b)

$$T = (T_t + T_a)N - T_a$$

$$E[T] = (T_t + T_a)E[N] - T_a = \frac{T_t + T_a}{1-q} - T_a.$$

(c)

$$T = (T_t + T_a)(N_1 + N_2) - 2T_a$$

$$E[T] = (T_t + T_a)(E[N_1] + E[N_2]) - 2T_a = \frac{2(T_t + T_a)}{1-q} - 2T_a.$$