MVE136 Random Signals Analysis – Test Exam

Aids: Beta or 2 sheets (4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

Grades: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

Good luck!

Task 1. Let (X, Y) be a continuous random variable with PDF $f_{X,Y}(x, y) = \mathrm{e}^{-x-y-xy} \,/\big(\int_0^\infty (1+z)^{-1} \mathrm{e}^{-z}\, dz\big)$ for $x, y \geq 0$ (and 0 otherwise). Find $\mathbf{E}\{X \,|\, Y = y\}$. **(5 points)**

Task 2. Find the probability $Pr(X(1)+X(2)+X(3) > 6)$ for a continuous time WSS Gaussian process $X(t)$ with mean $\mu_X = 1$ and autocorrelation function $R_{XX}(\tau) = \mathrm{e}^{-|\tau|} + 1$ for $\tau \in \mathbb{R}$. **(5 points)**

Task 3. Consider a discrete time Markov chain $X(n)$ with state space E and transition probability matrix P given by

$$E = \{0, 1\} \qquad \text{and} \qquad P = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix},$$

respectively. What initial distribution $\pi(0)$ of the chain will give a distribution $\pi(2)$ of the value of the chain $X(2)$ at time $n = 2$ given by $\pi(2) = [1/3 \;\; 2/3]$? **(5 points)**

Task 4. The PSD $S_{XX}(f)$ of a continuous time WSS process $X(t)$ has the properties to be real and symmetric (=even). Prove one of these properties. **(5 points)**

Task 5. A WSS discrete-time random process $X(n)$ with PSD $S_{XX}(f)$ is input to two different LTI systems with transfer functions $H_1(f)$ and $H_2(f)$, respectively. Find the cross spectral density $S_{Y_1Y_2}(f)$ between the outputs $Y_1(n)$ and $Y_2(n)$ from the two LTI systems. **(5 points)**

Task 6. Compute the autocorrelation function $r_x[n]$ for $n = 0$ and $n = 1$ when $x[n]$ is an AR(1)-process with parameter $a_1 = 0.7$. You can assume that the input noise has variance $\sigma_e^2 = 1$. **(5 points)**

MVE136 Random Signals Analysis – Solutions to Test Exam

Task 1. As $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^{\infty} \mathrm{e}^{-x-y-xy}\,dx / \big(\int_0^{\infty}(1+z)^{-1}\,\mathrm{e}^{-z}\,dz\big) = (1+y)^{-1}\,\mathrm{e}^{-y} / \big(\int_0^{\infty}(1+z)^{-1}\,\mathrm{e}^{-z}\,dz\big)$ for $y \geq 0$ (and 0 otherwise) we have $f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y) = (1+y)\,\mathrm{e}^{-x-xy}$ for $x, y \geq 0$ (and 0 otherwise), so that $\mathbf{E}\{X\,|\,Y=y\} = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx = \int_0^{\infty} x\,(1+y)\,\mathrm{e}^{-x(1+y)}\,dx = \ldots = (1+y)^{-1}$ for $y \geq 0$.

Task 2. As $X(1)+X(2)+X(3)$ is $N(m,\sigma^2)$-distributed we have $Pr(X(1)+X(2)+X(3) > 6) = Pr(N(m,\sigma^2) > 6) = 1 - \Phi((6-m)/\sigma)$, where $m = E[X(1)+X(2)+X(3)] = 3$ and $\sigma^2 = \mathrm{Var}(X(1)+X(2)+X(3)) = 3\,C_{XX}(0) + 4\,C_{XX}(1) + 2\,C_{XX}(2) = 3 + 4\,\mathrm{e}^{-1} + 2\,\mathrm{e}^{-2}$ [using that $C_{XX}(\tau) = R_{XX}(\tau) - \mu_X^2 = \mathrm{e}^{-|\tau|}$].

Task 3. Writing $\pi(0) = [p \;\; 1-p]$, we have

$$[1/3 \;\; 2/3] = \pi(2) = \pi(0)\,P^2 = [p \;\; 1-p]\begin{bmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{bmatrix} = [\frac{5}{16} + \frac{p}{16} \;\; \frac{11}{16} - \frac{p}{16}] \Leftrightarrow p = \frac{1}{3}.$$

Task 4. As the autocorrelation function $R_{XX}(\tau)$ is symmetric we have $S_{XX}(-f) = \int_{-\infty}^{\infty} R_{XX}(\tau)\,\mathrm{e}^{-j2\pi(-f)\tau}\,d\tau = \int_{-\infty}^{\infty} R_{XX}(\tau)\,\mathrm{e}^{-j2\pi f(-\tau)}\,d\tau = \int_{-\infty}^{\infty} R_{XX}(-\hat{\tau})\,\mathrm{e}^{-j2\pi f\hat{\tau}}\,d\hat{\tau} = \int_{-\infty}^{\infty} R_{XX}(\hat{\tau})\,\mathrm{e}^{-j2\pi f\hat{\tau}}\,d\hat{\tau} = S_{XX}(f)$ and $\overline{S_{XX}(f)} = \overline{\int_{-\infty}^{\infty} R_{XX}(\tau)\,\mathrm{e}^{-j2\pi f\tau}\,d\tau} = \int_{-\infty}^{\infty} R_{XX}(\tau)\,\mathrm{e}^{j2\pi f\tau}\,d\tau =$ [see above] $= S_{XX}(f)$.

Task 5. We have $S_{Y_1Y_2}(f) = \sum_{\tau=-\infty}^{\infty} \mathrm{e}^{-j2\pi f\tau}\,E[\sum_{k=-\infty}^{\infty} h_1(k)X(n-k) \sum_{\ell=-\infty}^{\infty} h_2(\ell)\, X(n+\tau-\ell)] = \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} \mathrm{e}^{-j2\pi f(\ell-k)} h_1(k)h_2(\ell) \sum_{\tau=-\infty}^{\infty} \mathrm{e}^{-j2\pi f(\tau-\ell+k)} R_{XX}(\tau-\ell+k) = \overline{H_1(f)}H_2(f)S_{XX}(f)$.

Task 6. We can use the Yule-Walker (YW) equations to find $r_x[0]$ and $r_x[1]$. Since it is simple, this solution will start with a derivation of the YW-equations: If we multiply both sides of the equation

$$x[n] + 0.7x[n-1] = e[n]$$

with $x[n-k]$ and take expectations we get

$$\underbrace{E\{x[n-k](x[n] + 0.7x[n-1])\}}_{r_x[k] + 0.7r_x[k-1]} = \underbrace{E\{x[n-k]e[n]\}}_{\delta[k]} \quad \text{for } k \geq 0.$$

Using this equation for $k = 0$ and $k = 1$ we get the matrix equation

$$\begin{bmatrix} 1 & 0.7 \\ 0.7 & 1 \end{bmatrix} \begin{bmatrix} r_x[0] \\ r_x[1] \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solving this matrix equation gives $r_x[0] = 1/0.51 \approx 1.96$ and $r_x[1] = -0.7r_x[0] \approx -1.37$.