# MVE136 Random Signals Analysis

# Written home exam Friday 30 October 2020 8.30–12.30

TEACHER AND EXAMINER: Patrik Albin 031 7723512 palbin@chalmers.se .

AIDS: All aids are permitted. (See the Canvas course "Ordinarie tentamen Modul: 0111, MVE136" with instructions for this exam for clarifications.)

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

**Task 1.** Calculate $\Pr(X(1)Y(2)Z(3) = 4)$ when $X(t)$, $Y(t)$ and $Z(t)$ are independent Poisson processes with intensity/rate 1. **(5 points)**

**Task 2.** Let $\{X(t)\}_{t\geq 0}$ be a zero-mean Gaussian process with autocorrelation function $R_{XX}(s,t) = \min(s,t)$. What probability distribution has $Y = X(1) + \int_0^1 X(s)\,ds$?

**(5 points)**

**Task 3.** Let $\{X_k\}_{k=0}^{\infty}$ be a Markov chain with state space $\{0,1,2\}$, initial distribution $\pi(0) = (1\ 0\ 0)$ and all transition probabilitites $p_{ij} = 1/3$. Calculate $\mathbf{E}\{T\}$ for $T = \min\{k > 0 : X_k = 2\}$. **(5 points)**

**Task 4.** A continuous time LTI system with insignal $X(t)$ has outsignal $Y(t) = \int_{-\infty}^{\infty} X(t-s)\,\frac{1}{1+s^2}\,ds$. What is the transfer function $H(f)$ of the system? **(5 points)**

**Task 5.** Let $Y(t) = \mathrm{e}^{-t/2}X(\mathrm{e}^t)$ for $t \in \mathbb{R}$ where $X(t)$ is the process in Task 2. Show that $Y(t)$ is a stationary process. **(5 points)**

**Task 6.** Explain how an AR(1) process $\{X_k\}_{k=-\infty}^{\infty}$ can be viewed as the limit of an $n$'th order MA process [i.e., MA($n$) process] as $n \to \infty$.

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## Solutions to written exam 30 October 2020

**Task 1.** The possible values of $(X(1), Y(2), Z(3))$ are $(4,1,1)$, $(1,4,1)$, $(1,1,4)$, $(2,2,1)$, $(2,1,2)$ and $(1,2,2)$ with probability $[\frac{1}{24}(1^4 \cdot 2 \cdot 3 + 1 \cdot 2^4 \cdot 3 + 1 \cdot 2 \cdot 3^4) + \frac{1}{4}(1^2 \cdot 2^2 \cdot 3 + 1^2 \cdot 2 \cdot 3^2 + 1 \cdot 2^2 \cdot 3^2)]\,\mathrm{e}^{-6} = \frac{51}{2}\,\mathrm{e}^{-6}$.

**Task 2.** Clearly, $Y$ is $\mathrm{N}(0,\sigma^2)$ with $\sigma^2 = \mathbf{E}\{X(1)^2\} + 2\,\mathbf{E}\{X(1)\,[\int_0^1 X(s)\,ds]\} + \mathbf{E}\{[\int_0^1 X(s)\,ds]^2\} = 1 + 2\int_0^1 R_{XX}(1,s)\,ds + \int_0^1\int_0^1 R_{XX}(s,t)\,dsdt = \ldots = \frac{7}{3}$.

**Task 3.** On the average it takes the chain the mean $3/2$ of a waiting time distribution with $p = 2/3$ to move from state 0. After this move with probability $1/2$ the chain is at state 1 and with probability is at state 2. Hence $\mathbf{E}\{T\} = 3/2 + (1/2)\cdot\mathbf{E}\{T\} + (1/2)\cdot 0$ giving $\mathbf{E}\{T\} = 3$.

**Task 4.** $H(f) = \pi\,\mathrm{e}^{-2\pi|f|}$.

**Task 5.** As $Y(t)$ is Gaussian it is enough to show that $Y(t)$ is WSS. Here $\mathbf{E}\{Y(t)\} = 0$ is constant while $R_{YY}(t,t+\tau) = \mathrm{e}^{-(t+t+\tau)/2}\min(\mathrm{e}^t, \mathrm{e}^{t+\tau}) = \mathrm{e}^{-\tau/2}$ for $\tau \geq 0$ and similary $R_{YY}(t,t+\tau) = \mathrm{e}^{\tau/2}$ for $\tau \leq 0$ do not depend on $t$.

**Task 6.** $X_k = e_k - aX_{k-1} = e_k - a\,e_{k-1} + a^2 X_{k-2} = \ldots = \sum_{i=0}^{n-1}(-a)^i e_{k-i} + (-a)^n X_{k-n} \to \sum_{i=0}^{\infty}(-a)^i e_{k-i}$ as $n \to \infty$ when $|a| < 1$.