MVE136 Random Signals Analysis

Written exam Monday 3 January 2022 2–6 PM

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AIDS: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Let Y_1, Y_2, Y_3 and Y_4 be independent standard normal random variables (with zero expectation and unit variance) and put $X_1 = (Y_1+Y_2)/\sqrt{2}$, $X_2 = (Y_1+Y_2+Y_3+Y_4)/2$ and $X_3 = (Y_1-Y_2)/\sqrt{2}$. State with reason if the following is true:

(a) X_1, X_2 and X_3 are all standard normal random variables. **(1,25 points)**

(b) X_i, $i=1,2,3$, is a Gaussian random process. **(1,25 points)**

(c) X_1 and X_2 are independent random variables. **(1,25 points)**

(d) X_1 and X_3 are independent random variables. **(1,25 points)**

Task 2. Let $\{X_k\}_{k=0}^{\infty}$ be a (time homogeneous) Markov chain. Is it true that $\mathbf{P}\{X_k = i_k | X_{k+1} = i_{k+1}, \ldots, X_{k+n} = i_{k+n}\} = \mathbf{P}\{X_k = i_k | X_{k+1} = i_{k+1}\}$ for $n \geq 1$? **(5 points)**

Task 3. Find a WSS discrete time random process $\{X(t)\}_{t\in\mathbb{Z}}$ that has autocorrelation function $R_{XX}(0)=3$, $R_{XX}(\pm 1)=2$, $R_{XX}(\pm 2)=1$ and $R_{XX}(\pm n)=0$ for $n\geq 3$. **(5 points)**

Task 4. Let T_1 and T_2 be the times for the first and second jump of a Poisson process $\{X(t)\}_{t\geq 0}$ with rate $\lambda > 0$. Find the joint probability density function of T_1 and T_2. HINT: $\mathbf{P}\{T_1 \leq s, T_2 \leq t\} = \mathbf{P}\{X(s)\geq 1,\, X(t)\geq 2\}$. **(5 points)**

Task 5. Let $\{X(t)\}_{t\in T}$ be a random process with autocorrelation function $R_{XX} : T\times T \to \mathbb{R}$. Is it true that $\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j R_{XX}(t_i,t_j) \geq 0$ for all $a_1,\ldots,a_n \in \mathbb{R}$, $t_1,\ldots,t_n \in T$ and $n\in\mathbb{N}$? **(5 points)**

Task 6. As you know the Wiener filter formula for filtration of a noise disturbed signal $X(t) = Z(t) + N(t)$ in order to optimally reconstruct $Z(t)$ is $H(f) = \frac{S_{ZZ}(f)}{S_{ZZ}(f)+S_{NN}(f)}$ when the signal $Z(t)$ and the noise $N(t)$ are independent and zero-mean WSS. How does this formula change if $Z(t)$ and $N(t)$ are dependent and jointly zero-mean WSS with crosspectraldensity $S_{ZN}(f)$? **(5 points)**

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Solutions to written exam 3 January 2022

Task 1. **(a)** As all X_i's are linear combination of independent normal random variables they are normal distributed. Calculations of means and variances give zero and one.

(b) All linear combination of the X_i's are linear combination of the independent normal distributed Y_i's and therefore are normal distributed. Hence X_i, $i = 1, 2, 3$, is Gaussian.

(c) X_1 and X_2 are not independent because $\mathbf{Cov}\{X_1, X_2\} = 1/\sqrt{2} \neq 0$.

(d) X_1 and X_3 are independent because they are jointly normal with $\mathbf{Cov}\{X_1, X_3\} = 0$.

Task 2. $\mathbf{P}\{X_k = i_k | X_{k+1} = i_{k+1}, \ldots, X_{k+n} = i_{k+n}\} = \frac{\mathbf{P}\{X_k = i_k\}\, p_{i_k, i_{k+1}} p_{i_{k+1}, i_{k+2}} \cdots p_{i_{k+n-1}, i_{k+n}}}{\mathbf{P}\{X_{k+1} = i_{k+1}\}\, p_{i_{k+1}, i_{k+2}} \cdots p_{i_{k+n-1}, i_{k+n}}}$ $= \frac{\mathbf{P}\{X_k = i_k\}\, p_{i_k, i_{k+1}}}{\mathbf{P}\{X_{k+1} = i_{k+1}\}} = \mathbf{P}\{X_k = i_k | X_{k+1} = i_{k+1}\}$ so yes.

Task 3. An MA(3)-process $X(t) = e_t + e_{t-1} + e_{t-2}$ where $\{e_t\}_{t \in \mathbb{Z}}$ is zero-mean and unit variance discrete time white noise.

Task 4. $f_{T_1, T_2}(s, t) = \frac{\partial^2}{\partial s \partial t} \mathbf{P}\{X(s) \geq 1,\ X(t) \geq 2\} = \frac{\partial^2}{\partial s \partial t} \mathbf{P}\{X(s) = 1\}\, \mathbf{P}\{X(t-s) \geq 1\} + \frac{\partial^2}{\partial s \partial t} \mathbf{P}\{X(s) \geq 2\} = \ldots = \lambda^2 \mathrm{e}^{-\lambda t}$ for $0 \leq s \leq t$.

Task 5. Yes because $\sum_{i=1}^n \sum_{j=1}^n a_i a_j R_{XX}(t_i, t_j) = \mathbf{E}[(\sum_{i=1}^n a_i X(t_i))^2]$.

Task 6. A straightforward modification of the derivation of the Wiener filter gives $H(f) = \frac{S_{ZZ}(f) + S_{ZN}(f)}{S_{ZZ}(f) + S_{NN}(f) + 2S_{ZN}(f)}$.