Tutorial 1 Problems

for discussion on Week 44, 2011

Sets, countability, events and σ-algebras (σ-fields)

1. Prove that a union of two countable sets is countable. Optionally, prove that a union of countably many of countable sets is countable.
2. Describe the sample space Ω for the following experiments:
   (a) Toss of 3 indistinguishable coins;
   (b) 3 tosses of one coin;
   (c) Drawing 2 balls from an urn containing 2 black and 2 white balls;
   (d) Picking a point at random from $[0, 1]$;
   (e) Picking a random ark from a circle (give at least two ways of defining Ω!)
3. Prove that σ-field $\mathcal{F}$ contains unions of any two of its sets. More generally, it contains at most countable unions of its sets.
4. Deduct from this that it also contains the symmetric differences $A\Delta B$ ('either A or B, but not both') for all $A, B \in \mathcal{F}$.
5. Prove Problem 1.2.1 from GS [1], (i.e. Problem 1 for Section 1.2, page 4). One of these identities has been shown at the lecture for the case of two sets (i.e. when $I = 1, 2$). Prove the remaining one for this case and, optionally, prove for a general at most countable I (i.e. finite or infinite countable)
6. Problem 1.2.4 from GS

Probability and its main properties

1. A cube with all its faces coloured is cut into 1000 equal size small cubes. Find the probability that a small ball chosen at random has exactly 2 coloured faces.

[1] In what follows, GS stands for the course book: Geoffrey Grimmett and David Stirzaker, *Probability and Random Processes*, Oxford University Press, 3rd edition, 2001. Problem x.y.z means Problem z for Section x.y in this book

2. Prove monotonicity of the probability: $P(A) \leq P(B)$ whenever $A \subseteq B$, $A, B \in \mathcal{F}$.

3. Derive formula for the probability of the symmetric difference of two events $A \Delta B$ in terms of their probabilities and the probability of their intersection.

4. Problem 1.3.1 from GS, page 8

5. Prove *inclusion–exclusion formula* from Problem 1.3.4 from GS, for $n = 3$ first. Try for general n by induction.

6. Problem 1.7.4 from GS.[2]

Symmetric experiment and combinatorics

1. Problem 1.8.1 from GS

2. The Ruritanian lottery requires that you pick a combination of 5 numbers from 50.

   (a) What is your chance of winning (one in ...)?

   (b) What is your chance of having no number correct?

   (c) What is your chance of having precisely one number correct?

[2] Notice that often a correct mathematical formulation of the problem is half its solution! Imagine an experiment when someone asks you every morning if you prefer a cup of coffee (x) to tea (y), and presumably your answer may depend on whether you slept well tonight (C). Denote A an event that at a randomly selected day you prefer coffee to tea. Is $\mathbf{P}(A) = 1$ under the given conditions?