Tutorial 2 Problems

for discussion on Week 45

Continuous spaces, Borel σ-field, density

1. Represent a closed segment on the line $[a, b]$ in terms of set-operations on open intervals. This will then yields that the Borel σ-field can be equally generated by the system of closed intervals.

2. A σ-field is called *countably generated* if it can be generated by a countable family of sets. Show that the system of open intervals on the line with rational end points is a countable family and that it generates the Borel σ-field $\mathcal{B}$. Thus $\mathcal{B}$ is countably generated.

3. The density of a probability measure $\mathbf{P}$ on the line is a constant c on the interval $[a, b]$ and 0 anywhere else. Find this constant and evaluate the measure $\mathbf{P}$ of $[a, (a+b)/2]$, $(a, (a+b)/2)$ and $[a+(b-a)/4, b-(b-a)/4]$.

4. The density of a probability measure $\mathbf{P}$ on the line has the form ax for $x \in [0, 5]$ and 0 anywhere else. Find the constant a and find the *median* of $\mathbf{P}$ which is such a point m that $\mathbf{P}(-\infty, m] = 1/2$[1] and evaluate the measure $\mathbf{P}$ of $[0, 2.5]$, $(2.5, 5)$.

5. This example shows that the sample space is not defined uniquely: it can always be enriched with null-sets. Consider rolling a symmetric die and the sample space $\Omega = \mathbb{R}_+ = [0, +\infty)$. Call $A = \{1, 2, 3, 4, 5, 6\}$ and define the probability measure $\mathbf{P}$ on the Borel sets $\mathcal{B}$ as follows:

$$\mathbf{P}(B) = \mathrm{card}(B \cap A)/6,$$

where $\mathrm{card}(B)$ for a finite set is the number of elements it has (note that $B \cap A$ is a finite set whatever B is!). In particular, taking $B = \{k\}$, we have $\mathbf{P}(k) = 1/6$ for $k = 1, \ldots, 6$, as expected. So the measure $\mathbf{P}$ is *atomic* with atoms of mass $1/6$ at points of A. Find now $\mathbf{P}(-\infty, 2.5]$ which is the probability that the number the die shows does not exceed 2.5. Is the result as expected?

[1] More exactly, the median is $\sup\{t : \mathbf{P}(-\infty, t] \leq 1/2\}$ which is the same as given above for measures with a density

Conditional, Full probability and Bayes formula

1. What is the probability that the sum of numbers shown on two symmetric dice is 8 if we know that the sum is even? The same question, given the sum is odd.
2. Problem 1.7.1 from GS
3. There are N lottery tickets lying on the table, among them $n < N$ are winning. The first person comes and picks one ticket at random, followed by the second and the third. Who has the largest chance of getting a winning ticket: the first, second or third person?
4. Three participants A, B and C of the shooting competition have the corresponding probabilities 3/5, 1/2 and 2/5 to hit the target, the data are based on their previous performance. They all fire at the same time at a single target and it appears that two bullets hit the target. What is more likely: competitor C hit or missed the target?

Product spaces, independence Event A is *independent* of B (with $\mathbf{P}(B) > 0$) if $\mathbf{P}(A \mid B) = \mathbf{P}(A)$ so that the frequency of occurrence of A is unaffected by the occurrence of B. Since in this case $\mathbf{P}(AB) = \mathbf{P}(A \mid B)\mathbf{P}(B) = \mathbf{P}(A)\mathbf{P}(B)$, the latter identity is taken as a definition of (pairwise) independence which holds also for the case $\mathbf{P}(B) = 0$.

1. Show that if A is indepedenent of B then B is independent of A.
2. Problem 1.5.1 from GS[2]
3. Problem 1.8.7 from GS.
4. Problem 1.5.7 a)-c) from GS.
5. Show that if two events A and B cannot happen at the same time and their probabilities are non-zero, then they are dependent (i.e. not independent).

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[2]In what follows, GS stands for the course book: Geoffrey Grimmett and David Stirzaker, *Probability and Random Processes*, Oxford University Press, 3rd edition, 2001. Problem x.y.z means Problem z for Section x.y in this book