Tutorial 3 Problems

for discussion on Week 46

Random variables, p.m.f. and c.d.f.

1. A random variable ξ has equal chances to take the following 3 values: -1,0 and 1. Find its p.m.f., draw its c.d.f. and find p.m.f.'s for the following random variables:

   (a) $|\xi|$;

   (b) $\xi^2 + 1$;

   (c) 2^ξ;

   (d) $\xi_+ = \max\{\xi, 0\}$.

2. Finish the proof outlined at the lecture that the c.d.f. $F_\xi(x)$ possesses the following properties:

   (a) $F_\xi(x)$ is continuous from the right.

   (b) Derive the formulae:

$$\begin{aligned}
\mathbf{P}\{\xi \in (a,b]\} &= F_\xi(b) - F_\xi(a);\\
\mathbf{P}\{\xi \in [a,b]\} &= F_\xi(b) - F_\xi(a-);\\
\mathbf{P}\{\xi \in [a,b)\} &= F_\xi(b-) - F_\xi(a-);\\
\mathbf{P}\{\xi \in (a,b)\} &= F_\xi(b-) - F_\xi(a),
\end{aligned}$$

   where $F_\xi(x-) = \lim_{\varepsilon \downarrow 0} F_\xi(x - \varepsilon)$.

3. Problem 2.1.2 from GS.[1]

4. Find the c.d.f. of a random point chosen uniformly on a segment $[a, b]$.

5. A point is randomly uniformly thrown on a square with the sides parallel to the coordinate axes (i.e. both coordinates of the point are uniformly distributed). What is the probability that it lies:

   (a) in the upper left quarter of the square;

[1] In what follows, GS stands for the course book: Geoffrey Grimmett and David Stirzaker, *Probability and Random Processes*, Oxford University Press, 3rd edition, 2001. Problem x.y.z means Problem z for Section x.y in this book

(b) in a given rectangle inside the square;

(c) inside the circle circumscribed in the square?

6. Problem 2.7.4 from GS

7. Problem 2.3.3 from GS. This property is the basis of computer simulations of random variables known as Monte-Carlo methods.

Expectation of a random variable

1. Two male students A and B make a bet: A suggests that he would ask the first female student entering the auditorium for her telephone number and she will give it to him. If this happens, B will pay A 10 kronor. Otherwise, A promises to pay B 10 kronor. How much A is expecting to win (or loose) if he believes he's so attractive that $p \cdot 100\%$ ($p \in [0, 1]$) percent of girls would readily share a telephone with him? For which values of p he should not really launch himself in such a bet?

2. Let ξ be the *Bernoulli*-distributed r.v., i.e. the discrete r.v. taking values 0 and 1 only with respective probabilities $1 - p$ and p. Find $\mathbf{E}\,\xi$. Find its distribution, i.e. the measure $\mathbf{P}_\xi$ on the Borel σ-algebra $\mathcal{B}$ such that $\mathbf{P}_\xi(B) \stackrel{\text{def}}{=} \mathbf{P}(\xi^{-1}(B)) = \mathbf{P}\{\xi \in B\}$ for any Borel $B \in \mathcal{B}$.

3. Let ξ be an exponentially distributed random variable $\mathrm{Exp}(\lambda)$ with a parameter $\lambda > 0$, i.e. with the p.d.f.

$$f_\xi(x) = \begin{cases} 0 & \text{for } x < 0; \\ \lambda e^{-\lambda x} & \text{for } x \geq 0. \end{cases}$$

Find its c.d.f, expectation and variance.

4. Problem 3.3.4 from GS. By a 'fair fee' the authors mean the average gain you are expected to receive in this game, arguing that in this case your gain (as well as the gain of the game organiser) will be on average 0 after many repetitions of the game – quite fair indeed!

5. Problem 3.3.1 from GS (construct a contre-example/example of a discrete r.v. for each of both statements).