Tutorial 4 Problems

for discussion on Week 47

Lebesgue integral

1. Show that if $f(x) \geq 0$ for all x than the Lebesgue integral $\int f(x)\mu(dx)$ is non-negative.

2. Show that if $f(x) \geq g(x)$ for all x than also $\int f(x)\mu(dx) \geq \int g(x)\mu(dx)$.

3. Deduce from here that if $f(x) \geq 0$ for all x, then $\int f(x)\mu(dx) \geq \mu(B)\inf_B f(x)$ for any measurable B.

4. Let μ be a two-atom measure $\mu = (1-p)\delta_0 + p\delta_1$ on the Borel sets of $\mathbb{R}$, where δ_x stands for the unit mass measure concentrated at a point x: $\delta_x(B) = \mathbb{1}_B(x)$ for any Borel set $B \in \mathcal{B}$. Compute the Lebesgue integrals:

   (a) $\int x\mu(dx)$;
   (b) $\int x^2\mu(dx)$;
   (c) $\int 1\mu(dx)$;
   (d) $\int (x-p)^2\mu(dx)$.

   What random variable this distribution corresponds to?

Expectation and its properties

1. For a discrete r.v. ξ taking values $\{x_1, x_2, \dots\}$ find the corresponding distribution $\mathbf{P}_\xi$ on $[\mathbb{R}, \mathcal{B}]$ and check that

$$\mathbf{E}\,\xi = \int \xi(\omega)\,\mathbf{P}(d\omega) = \int x\,\mathbf{P}_\xi(dx)\,.$$

2. Show that $\mathbf{E}\,\xi_1\xi_2 = \mathbf{E}\,\xi_1\,\mathbf{E}\,\xi_2$ for *independent* random variables. As at the lecture, show this for discrete random variables and then use monotonicity argument for general ξ_i's.

3. Show that if for a random variable ξ such that $\mathbf{P}\{\xi \geq 0\} = 1$ [1] one also has $\mathbf{E}\,\xi = 0$, then $\mathbf{P}\{\xi = 0\} = 1$, i.e. $\xi = 0$ a.s. Deduce from here that if $\mathbf{var}\,\xi = 0$ then $\mathbf{P}\{\xi = c\} = 1$ for some constant c.

[1] We say in this case that $\xi \geq 0$ *almost surely* (abbreviated a.s.)

4. Show that if $\xi \geq \eta$ a.s. then $\mathbf{E}\,\xi \geq \mathbf{E}\,\eta$. Deduce from this that if $\xi = \eta$ a.s. (such random variables are called *equivalent*) then $\mathbf{E}\,\xi = \mathbf{E}\,\eta$.

5. The Russian government in 1961 carried out devaluation of the national currency: the new ruble in 1961 became equal 10 old rubbles. Assume an average salary of workers employed in a certain industry in Russia in 1961 before the monetary reform was 1400 rubles. What is expected salary of a randomly chosen worker after the reform? Prove your answer!

6. ξ is equally likely to take any values among $0, 1, \dots, n$ and η is equally likely to take any values among $-m, -m+1, \dots, 0, \dots, m$.

   (a) What is $\mathbf{E}(\xi + \eta)$?

   (b) Compute $\mathbf{E}\,\xi\eta$ and $\mathbf{E}\,|\xi\eta|$ if we know that ξ and η are independent.

7. For a r.v. ξ, introduce two random variables $\xi_+ = \max\{0, \xi\}$ and $\xi_- = \min\{0, \xi\}$. What is $\mathbf{E}(\xi_+ + \xi_-)$ and $\mathbf{E}(\xi_+ - \xi_-)$?

8. Problem 3.11.1 from GS[2]

[2]As usual, GS stands for the course book: Geoffrey Grimmett and David Stirzaker, *Probability and Random Processes*, Oxford University Press, 3rd edition, 2001. Problem x.y.z means Problem z for Section x.y in this book