Tutorial 6 Problems

for discussion on Week 49

Multivariate distributions

1. Let (ξ_1, ξ_2) be a point uniformly distributed in the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ in the plane. Find

   (a) the marginal distribution of ξ_1;

   (b) the mean vector of (ξ_1, ξ_2);

   (c) the covariance matrix.

   Are ξ_1 and ξ_2 independent?

2. Prove that if both components in a two-dimensional multivariate Normal random vector are uncorrelated then they are independent (this is NOT true in general for non-Normal r.v.'s!). *Hint:* How does the covariance matrix look like in this case?

3. Let ξ, η are two independent standard Normal random variables. Find the c.d.f. of the distance from the point (ξ, η) in the plane to the origin. This distribution is called *Raleigh distribution*. *Hint:* pass to the polar coordinates in the integral of the joint p.d.f.

4. In the settings of the previous task, find the distribution of the polar angle of the vector (ξ, η).

Conditional distributions and sums

1. Work out to the end the Example (5) on p.68 of GS[1]

2. Problem 3.7.1(a-e) from GS.

3. Problem 3.11.4 from GS

4. Problem 3.11.8 from GS

[1] As usual, GS stands for the course book: Geoffrey Grimmett and David Stirzaker, *Probability and Random Processes*, Oxford University Press, 3rd edition, 2001. Problem x.y.z means Problem z for Section x.y in this book

5. Problem 3.11.6a from GS

6. Prove the *lack of memory* property of the Exponential distribution:

$$\mathbf{P}\{\xi > x + y \mid \xi > x\} = \mathbf{P}\{\xi > y\}\,.$$

7. Find the distribution of the sum of two independent uniformly distributed in $[0, 1]$ random variables. Is it uniform?

8. A random point (ξ, η) is uniformly distributed in the triangle $\{(x, y) : x + y \leq 1,\ x, y \geq 0\}$. Find:

   (a) the conditional density of ξ given $\eta = y$. What distribution is that?

   (b) the conditional expectation $\mathbf{E}[\xi \mid \eta]$, its expectation and its variance.

   (c) covariance $\mathbf{cov}(\xi, \eta)$.

Main inequalities and Laws of Large Numbers

1. Establish inequality

$$\mathbf{P}\{|\xi - \mathbf{E}\,\xi| \geq \varepsilon\} \leq \frac{\mathbf{E}(\xi - \mathbf{E}\,\xi)^4}{\varepsilon^4}$$

   and use it to prove the Strong Law of large Numbers started at the lecture: $\varphi_n = \frac{1}{n}\sum_{k=1}^{n}\chi_k$ converges almost surely to $\mathbf{P}(A)$, where χ_k is the indicator of the event A in the kth independent trial. It was shown at the last lecture that for this it is sufficient that $\mathbf{P}\{|\varphi_n - \mathbf{P}(A)| > 1/k\ i.o.\} = 0$ for all k. Use the Borel-Cantelli lemma to show this.

2. Establish the Chernoff bound[2] explicitly for random variables having the following distributions:

   (a) Bernoulli;

   (b) Poisson;

   (c) Exponential;

   (d) Standard Normal.

[2]Recall, the Chernoff bound or the Exponential Chebyshov inequality is: $\mathbf{P}\{\xi \geq \varepsilon\} \leq \inf_{\lambda>0} e^{-\lambda\varepsilon}\,\mathbf{E}\,e^{\lambda\xi}$