Tutorial 7 Problems

for discussion on Week 50 at Consultation session on Monday 13:15

Different types of convergence and Borel-Cantelli lemma

1. Establish the following Chernoff bound: there exists $c_2 > 0$ such that $\mathbf{P}\{\varphi_n - np < -n\varepsilon\} < e^{-c_2 n}$, where φ_n is the frequency of an event having probability p in n independent experiments. Hint: see how it was done to arrive to $\mathbf{P}\{\varphi_n - np > n\varepsilon\} < e^{-c_1 n}$ at the lecture.

2. Let $\{\xi_n\}$ be a sequence of r.v.'s defined on the same probability space. Prove that their a.s. convergence implies convergence in probability. Hint: it was shown at the last lecture that $\mathbf{P}\{\xi_n \not\to \xi\} = \lim_{k\to\infty} \mathbf{P}\{|\xi_n - \xi| > 1/k \ i.o.\} = 0$ if $\xi_n \to \xi$ a.s. Relate the last probability to $\mathbf{P}\{|\xi_n - \xi| > 1/k\}$.

3. Understand the following contre-example which shows that convergence in probability does *not* imply a.s. convergence. Consider infinite number (indexed by $m = 1, 2, \dots$) of finite series (series m is indexed by $k_m = 1, \dots, m$) of random variables on the probability space $([0,1], \mathcal{B}, \ell)$: $\eta_{k_m,m}(\omega) = \mathbb{1}_{[(k_m-1)/m, k_m/m]}(\omega)$. Then let ξ_1 be $\eta_{1,1}$, $\xi_2 = \eta_{1,2}$, $\xi_3 = \eta_{2,2}$, $\xi_4 = \xi_{1,3}$, $\xi_5 = \xi_{2,3}$, etc. Show that $\xi_n \to 0$ in probability, but ξ_n does not converge a.s.

4. Problem 7.3.12 from GS[1]

Weak convergence

1. Let $\{\xi_n\}$ be a sequence of r.v.'s on the same probability space converging in probability: $\xi_n \xrightarrow{\mathbf{P}} \xi$. Then it also converges weakly: $\xi_n \xrightarrow{w} \xi$. Understand the proof of this fact given as Lemma 5 of Section 7.2 in GS.

2. Let $\{\xi_n\}$ be a sequence of r.v.'s on the same probability space weakly converging to a constant: $\xi_n \xrightarrow{w} c$. Show that then it converges also in probability: $\xi_n \xrightarrow{\mathbf{P}} c$.

[1] As usual, GS stands for the course book: Geoffrey Grimmett and David Stirzaker, *Probability and Random Processes*, Oxford University Press, 3rd edition, 2001. Problem x.y.z means Problem z for Section x.y in this book

Characteristic functions and the CLT

1. Compute the characteristic functions for the following distributions:
   (a) Bernoulli with parameter p;
   (b) Binomial $\mathrm{Binom}(n, p)$ (use the previous result!);
   (c) Uniform distribution on $[0, 1]$;
   (d) Uniform distribution on $[a, b]$. When is it real-valued?

2. Show that the characteristic function is real if and only if the distribution is symmetric, i.e. the random variable $-\xi$ is distributed as ξ.

3. Prove that a characteristic function is positive semi-definite, i.e.

$$\sum_{j,k} \varphi(t_k - t_j) s_j \overline{s_k} \geq 0$$

   for any real $t_1, \dots, t_n \in \mathbb{R}$ and complex numbers $s_1, \dots, s_j \in \mathbb{C}$ ($\overline{s}$ for a complex number $s = x + iy$ denotes its complex conjugate $x - iy$).

4. Prove the Poisson limit theorem with the help of characteristic functions.