Lektion 4

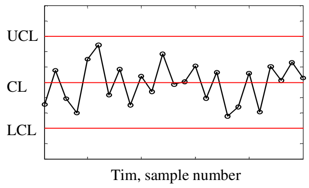
2007-11-21

Chapter 8

CUMSUM and EWMA

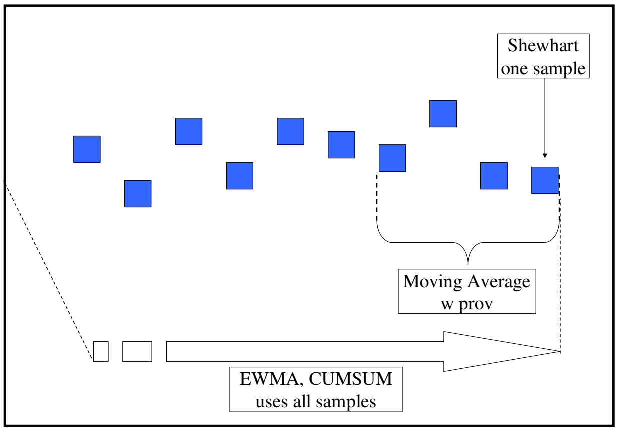
Shewhart

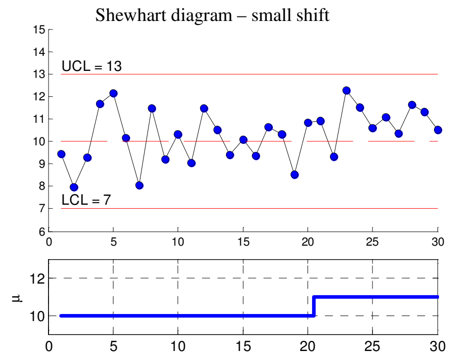
- Good in Phase I!
- Detects large shifts.
- Not sensitive for small shifts.
- Only information from the sample.
- Sensitizing rules – decreases ARL.



Three diagrams for Phase II

- **CumSum** (The Cumulative Sum Control Chart)
- **EWMA** (The Exponentially Weighted Moving Average Control Chart)
- **Glidande medelvärde** (The Moving Average Chart)
- Good at detecting small shifts in average.
- μ and σ are often supposed to be known.





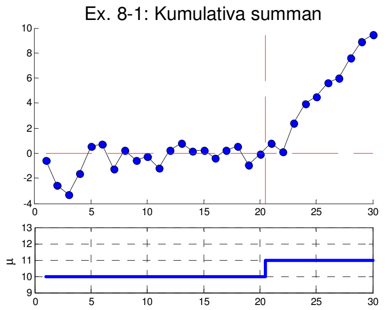
Exemple 8-1

- Shewhart failed to detect the small shift.
- Study the Cumulative Sum instead.

$$C_i = \sum_{j=1}^{i}(x_j - \mu_0) =$$
$$= (x_i - \mu_0) + \sum_{j=1}^{i-1}(x_j - \mu_0) =$$
$$= (x_i - \mu_0) + C_{i-1}$$

Prov i	x_i	$x_i - 10$	C_i
1	9.45	−0.55	−0.55
2	7.99	−2.01	−2.56
3	9.29	−0.71	−3.27
4	11.66	1.66	−1.61
⋮	⋮	⋮	⋮
30	10.52	0.52	9.45

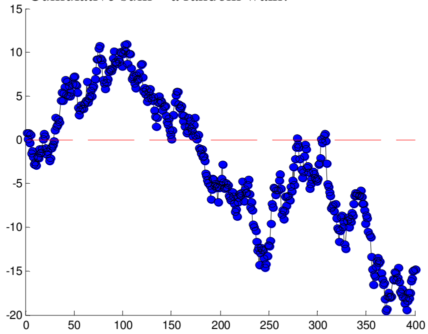
$$\begin{cases} \mu_0 = 10 \\ \sigma = 1 \end{cases}$$



- Lacks control limits!
- How alarm?
- V-masl
- Tabular-CumSum

C is a random walk! C is random walk with drift!

Cumulative sum – a random walk!



- Stable process
- Wiener process?

Tabular-CumSum

- Monitor average
- n=1
- x is N(μ_0,σ)
- μ_0 is target value.
  - Control a chemical process
- OCAP is of cause neccessary!

Tabular-Cumsum

The Tabular Cumsum

$C_i^+ = \max\left[0, x_i - (\mu_0 + K) + C_{i-1}^+\right]$

$C_i^- = \max\left[0, (\mu_0 - K) - x_i + C_{i-1}^-\right]$

Start values: $\begin{cases} C_0^+ = 0 \\ C_0^- = 0 \end{cases}$

Reference value: $K = \frac{\delta}{2}\sigma = \frac{|\mu_1 - \mu_0|}{2}$

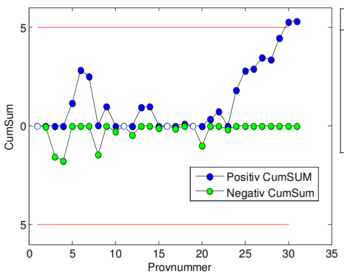
Alarm if $\begin{cases} C_i^+ > H = 5\sigma \\ C_i^- > H = 5\sigma \end{cases}$

Counter:

$N_i^{+} =$ Number of consecutive $C_i^+ > 0$

$N_i^{-} =$ Number of consecutive $C_i^- > 0$

Ex. 8-1 Tabular CuSum



$Period(i)$	x_i	C_i^+	N_i^+	C_i^-	N_i^-
1	9.45	0	0	0.05	1
2	7.99	0	0	1.56	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
28	11.62	4.47	6	0	0
29	11.31	5.28	7	0	0
30	10.52	5.30	8	0	0

Alarm at point 29: $C_{29}^+ = 5.28 > H = 5$

$N_{29}^+ = 7$

Something happened at time 29 -7=22!

What to do when Alarm?

- Find the cause! (OCAP)
  - Aid from the counter.
- Restart Cusum at 0 again.
- Calculate the new average:

$$\hat{\mu} = \begin{cases} \mu_0 + K + \frac{C_i^+}{N_i^+} & \text{if } C_i^+ > H \\ \mu_0 - K - \frac{C_i^-}{N_i^-} & \text{if } C_i^- > H \end{cases}$$

$$\hat{\mu} = \mu_0 + K + \frac{C_i^+}{N_i^+} = \\ = 10 + 0.5 + \frac{5.28}{7} = \\ = 11.25$$

H, K and ARL for CuSum

$$\text{Let } \begin{cases} H = h\sigma \\ K = k\sigma \end{cases}$$

ARL for $k = \frac{1}{2}$:

Skift in average (multiple of σ)	$h=4$	$h=5$
0	168	465
0.25	74.2	139
0.50	26.6	38
0.75	13.3	17
1.00	8.38	10.4
1.50	4.75	5.75
2.00	3.34	4.01
2.50	2.62	3.11
3.00	2.19	2.57
4.00	1.71	2.01

$ARL_0 = 370$

k	h
0.25	8.01
0.5	4.77
0.75	3.34
1.0	2.52
1.25	1.99
1.5	1.61

Calculation of ARL

- Markov chains
- Approximative method:

$$ARL^{(+-)} = \frac{\exp(-2\Delta b) + 2\Delta b - 1}{2\Delta^2}$$

$$\Delta = \delta^* - k \text{ för } C_i^+$$

$$\Delta = -\delta^* - k \text{ för } C_i^-$$

$$\delta^* = \frac{\mu_1 - \mu_2}{\sigma}$$

$$b = h + 1.166$$

$$\frac{1}{ARL} = \frac{1}{ARL^+} + \frac{1}{ARL^-}$$

Standardized CuSum

Let

$$y_i = \frac{x_i - \mu_0}{\sigma}$$

$$C_i^+ = \max\left[0, y_i - k - C_{i-1}^+\right]$$

$$C_i^- = \max\left[0, -y_i - k - C_{i-1}^-\right]$$

Advantages:
- h and k not dependent on σ.
- leads to Cusum for process variation.

Disadvantage:
- Loses the units.
- Less understanding.

V-mask

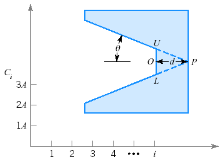


Figure 8-5 A typical V-mask.

The tabular cusum and the V-mask scheme are equivalent if

$$k = A \tan\theta \qquad (8\text{-}17)$$

and

$$h = A\, d \tan(\theta) = dk \qquad (8\text{-}18)$$

$$C_i = y_i + C_{i-1}$$

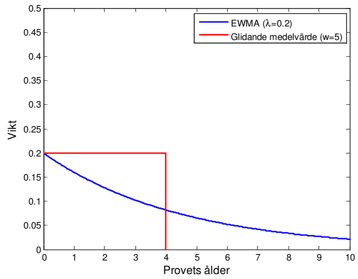
Warning:

1. Difficult to start V-mask
2. How long arms?
3. Choice of parameters.

EWMA
Exponentially Weighted Moving Average

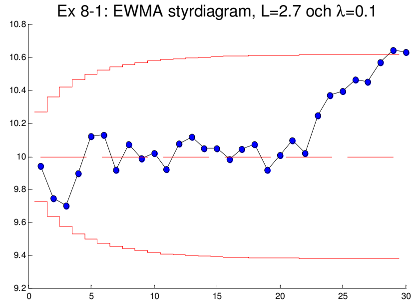
$$z_i = \lambda x_i + (1-\lambda) z_{i-1}$$
$$z_0 = \mu_0$$
$$0 < \lambda < 1$$

$$\mu_{z_i} = \mu_0$$
$$\sigma_{z_i}^2 = \sigma^2 \left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]$$



EWMA –control chart

$$UCL_i = \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2i}\right]} \rightarrow \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}}$$
$$CL_i = \mu_0$$
$$LCL_i = \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2i}\right]} \rightarrow \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}}$$



Alarm at time point 29!

Design of EWMA

- Efficient for small shifts in average.
- Robust against normal distribution assumption!
- Choose L and λ for wanted ARL.

Shift in Mean (multiple of σ)	$L = 3.054$ $\lambda = 0.40$	2.998 0.25	2.962 0.20	2.814 0.10	2.615 0.05
0	500	500	500	500	500
0.25	224	170	150	106	84.1
0.50	71.2	48.2	41.8	31.3	28.8
0.75	28.4	20.1	18.2	15.9	16.4
1.00	14.3	11.1	10.5	10.3	11.4
1.50	5.9	5.5	5.5	6.1	7.1
2.00	3.5	3.6	3.7	4.4	5.2
2.50	2.5	2.7	2.9	3.4	4.2
3.00	2.0	2.3	2.4	2.9	3.5
4.00	1.4	1.7	1.9	2.2	2.7

EWMA

- Works with sample groups
- Montitor variability
- Poisson data
- Forecasting process (z_i is a predictor)

Moving average
(Glidande medelvärde)

$$M_i = \frac{x_i + x_{i-1} + \ldots + x_{i-w+1}}{w}$$

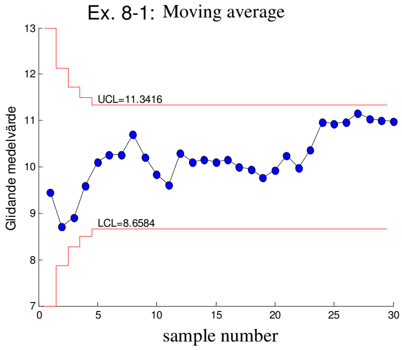
$$V(M_i) = \frac{1}{w^2} \sum_{j=i-w+1}^{i} V(x_j) = \frac{\sigma^2}{w^2}$$

Observe that M_i and M_j are correlated for $|i-j| \leq w$!

Control chart:

$$\begin{cases} UCL = \mu_0 + 3\dfrac{\sigma}{\sqrt{w}} \\ LCL = \mu_0 - 3\dfrac{\sigma}{\sqrt{w}} \end{cases}$$

Ex. 8-1: Moving average



OBS!
No alarm at time point 28!