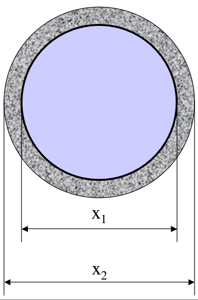
Lektion 6

2007-12-06_1
Chapter 10
Multivariate data

Multivariate Process Control
Chapter 10

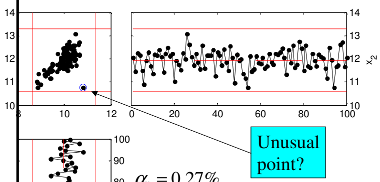
- Several interesting variables
- Look at all variables at the same time
- Hotellings T^2-diagram
  - Analogue to Shewhart
- EWMA
- Principal component analysis
  - Decreas the number of variables to control.

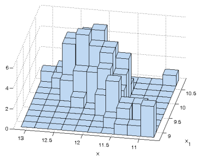
Example
Inner and outer diameter on a bearing

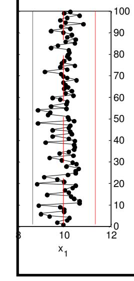


$X_1 \sim N(\mu_1, \sigma)$
$X_2 \sim N(\mu_2, \sigma)$

How should the process be controlled?







$$\alpha_1 = 0.27\%$$

$$\alpha_2 = 0.27\%$$

but

$$P(\text{Type I-error}) = \alpha' = 1-(1-\alpha)^p =$$

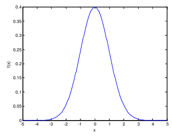
$$= 1-(1-0.0027)^2 = 0.54\%$$

$$ARL = \frac{1}{\alpha'} = 185$$

Normal multivariate data

1 dimension:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



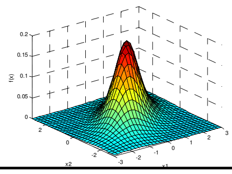
p dimensions:

$$f(x) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

$p =$ number of dimensions

$\mu = [\mu_1, \dots, \mu_p]$ is the average vector

Σ is the covariance matrix



Multivariate statistics

Suppose that

$x_1, x_2, \dots x_n$

is a sample from a p-dimensional stochastic variable.

where

$x_i = [x_{i1}, x_{i2}, \dots, x_{ip}]$

The sample average is

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

and the sample covariance is

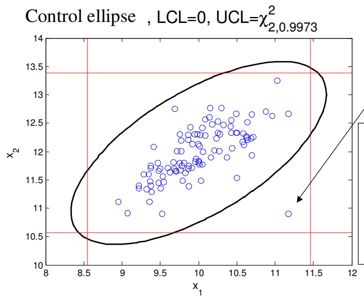
$$S = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})'$$

$$s_j^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_{ij} - \bar{x}_j)^2$$

$$s_{jk} = \frac{1}{n-1}\sum_{i=1}^{n}(x_{ij} - \bar{x}_j)(x_{ik} - \bar{x}_k)$$

$$\begin{cases} E(\bar{x}) = \mu \\ E(S) = \Sigma \end{cases}$$

Control chart - a first attempt



Alarm for sample 72!

Two disadvantages

1. Time order is lost.!
2. Not practicable or possible for more than 2 variables!

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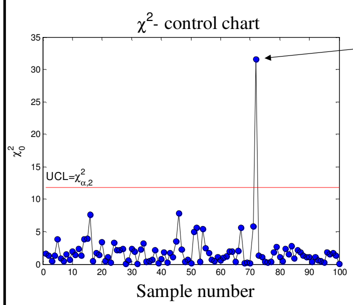
Draw a χ^2-diagram



Alarm for sample 72!

Generally for p variables:

$$\chi_0^2 = n(\bar{x}-\mu)'\Sigma^{-1}(\bar{x}-\mu)$$

$$\chi_0^2 \sim \chi_p^2 \text{ (p degrees of freedom)}$$

$$UCL = \chi^2_{1-\alpha,p}$$

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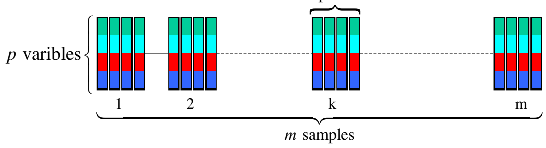
Estimate μ and Σ!

$$\begin{matrix}\bar{x}_{jk} \\ s_{jk}^2 \\ s_{jhk}\end{matrix} \quad \begin{cases} j=1,\ldots,p \\ k=1,\ldots,m \\ j\neq h\end{cases}$$



$$\bar{\bar{x}}_j = \frac{1}{m}\sum_{k=1}^{m}\bar{x}_{jk} \quad j=1,2,\ldots,p$$

$$\bar{s}_j^2 = \frac{1}{m}\sum_{k=1}^{m}s_{jk}^2 \quad j=1,2,\ldots,p$$

$$\bar{s}_{jh} = \frac{1}{m}\sum_{k=1}^{m}s_{jhk} \quad j\neq h$$

$$\bar{\bar{x}} = \begin{bmatrix}\bar{\bar{x}}_1 \\ \bar{\bar{x}}_2 \\ \vdots \\ \bar{\bar{x}}_p\end{bmatrix} \qquad S = \begin{bmatrix}\bar{s}_1^2 & \bar{s}_{12} & \cdots & \bar{s}_{1p} \\ & \bar{s}_2^2 & \cdots & \bar{s}_{2p} \\ & & \ddots & \vdots \\ & & & \bar{s}_p^2\end{bmatrix} \qquad \begin{cases}\hat{\mu} = \bar{\bar{x}} \\ \hat{\Sigma} = S\end{cases}$$

Hotellings T²-control chart

- Different control limits for Phase I and Phase II!

$$T^2 = n(\bar{x}-\bar{\bar{x}})' S^{-1} (\bar{x}-\bar{\bar{x}})$$

Phase I:

$$\begin{cases} UCL = \frac{p(m-1)(n-1)}{mn-m-p+1} F_{1-\alpha,p,mn-m-p+1} \\ LCL = 0 \end{cases}$$

Phase II:

$$\begin{cases} UCL = \frac{p(m+1)(n-1)}{mn-m-p+1} F_{1-\alpha,p,mn-m-p+1} \\ LCL = 0 \end{cases}$$

Interpretation of Alarm-signal

- Which variable is the cause of the alarm?
  - Could be difficult. See earlier example.
  - Draw individual control charts.
- Study

$$d_i = T^2 - T^2_{(i)}$$

$T^2_{(i)}$ is the test statistic for all vairables excluding variable *i*.

Is d_i large for some variable *i* then will this variable be a good candidate for further studies.

Hotellings T²-control charts Individual observations, n=1

- Different control limits for Phase I and Phase II!

$$T^2 = (\bar{x}-\bar{\bar{x}})' S^{-1} (\bar{x}-\bar{\bar{x}})$$

Phase I:

$$\begin{cases} UCL = \frac{(m-1)^2}{m} \beta_{1-\alpha,p/2,(m-p-1)/2} \\ LCL = 0 \end{cases}$$

Phase II:

$$\begin{cases} UCL = \frac{p(m+1)(m-1)}{m^2-mp} F_{1-\alpha,p,m-p} \\ LCL = 0 \end{cases}$$