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26 pages

Week 4) Statistical Process Control (SPC)

4a) How SPC Works

The magnificent seven.

1. Histogram or stem-and-leaf plot
2. Check sheet
3. Pareto chart
4. Cause-and-effect diagram
5. Defect concentration diagram
6. Scatter diagram
7. Control chart

Causes of quality variation.

Process is in statistical control if variation is due to chance only.

Process is out-of-control if if variation is due to assignable causes.

Main process parameters are

- the mean μ
- the standard deviation σ

In principle μ, σ are time dependent, and the process is in control as long as $\mu = \mu_0$ and $\sigma = \sigma_0$, where μ_0 and σ_0 are the in control values.

The process is out-of-control due to assignable causes if at some time point at least one of μ and σ have shifted to a new value.

When the process is in control, most of the production will fall between LSL and USL.

When the process is out-of-control, a higher proportion of the production lies outside of these specifications.

Is the goal of SPC to reduce process variability as much as possible, or is it to have control over unavoidable variability?

Basic principles of control charts.

The purpose of a control chart is to control the values of a process parameter. We shall consider $\bar{x}$ (mean), s (standard deviation) and R (range) charts.

For an $\bar{x}$ control chart, typically

$$\begin{aligned} \mathrm{UCL} &= \mu_0 + 3\sigma_0/\sqrt{n} \\ \mathrm{CL} &= \mu_0 \\ \mathrm{LCL} &= \mu_0 - 3\sigma_0/\sqrt{n} \end{aligned}$$

where n is the size of the control sample.

It is good if

$$\mathrm{USL} > \mathrm{UCL} \text{ and } \mathrm{LSL} < \mathrm{LCL}$$

but this is not always the case.

See Figure 5.2 on p 182.

Sometimes action or warning limits are defined. Typically,

$$\begin{aligned} \mathrm{UWL} &= \mu_0 + 2\sigma_0/\sqrt{n} \\ \mathrm{LWL} &= \mu_0 - 2\sigma_0/\sqrt{n} \end{aligned}$$

It is assumed that the control samples consist of i.i.d normal or nearly normal observations, and that consecutive samples are independent.

The most important use of a control chart is to improve the process. Read **1.**, **2.** and **3.** on p 185.

Sample size and sampling frequency.

The larger the sample size n is, the larger is the probability $1-\beta$ of detecting a small shift in the process.

The probability of a false alarm does not depend on the sample size. It is

$$\alpha = P\left(\left|\frac{\bar{x}-\mu_0}{\sigma_0/\sqrt{n}}\right| > 3\right) = 2(1-\Phi(3)) = 0.0027$$

The average run length (ARL) (unit is number of samples) depends on whether the process is in control or not. The in control value is

$$\mathrm{ARL}_0 = \frac{1}{\alpha} = \frac{1}{0.0027} = 370$$

The out-of-control value depends on how big the shift in the mean is.

Ideally, the control samples should be taken so that parameter shifts tend to occur between samples—not within. This is the idea behind the *rational subgroup* concept. See Section 5.3.4.

A slightly unorthodox example of the control chart methodology is regurlarly made surveys of political opinions from the viewpoint of a political party.

Analysis of control chart patterns.

See Western Electric Handbook suggestions on p 196.

Suppose k decision rules are used and that criterion i has type I error probability α_i, then the overall type I error probability is

$$\alpha = 1 - \prod_i (1 - \alpha_i) \quad \text{if the rules are independent}$$

In any case,

$$\alpha \leq \sum_i \alpha_i \quad \text{by Boole's inequality}$$

Phase I and Phase II applications of control charts.

In phase I, the purpose is to find the control limits.

In Phase II, the purpose is to produce items.

Reading hints.

Section 5.4 The Rest of the Magnificient Seven is left for self studies. So is also Section 5.5 Implementing SPC in a Quality Improvment Program, and Section 5.6 An Application of SPC.

Section 5.7 Applications of Statistical Process…may be omitted.

4b) Variables Control Charts

Statistical basis.

Typically the in control values of μ and σ are not known.

In order to estimate μ and σ, the recommendation is to take $m \gtrapprox 25$ samples, each of size $n \gtrapprox 5$.

The best estimator of μ is the grand mean

$$\bar{\bar{x}} = \frac{\bar{x}_1 + \cdots + \bar{x}_m}{m}$$

The best estimator of σ^2 is the pooled sample variance

$$s_P^2 = \frac{s_1^2 + \cdots + s_m^2}{m}$$

The grand mean and the pooled variance are unbiased estimators of their theoretical counterparts.

The average standard deviation is

$$\bar{s} = \frac{1}{m} \sum_i s_i$$

Note that

$$Es = c_4 \sigma \quad \text{and} \quad \operatorname{Var} s = (1 - c_4^2)\sigma^2$$

Let $R_1, \ldots, R_m$ be the ranges of the m samples. The average range is

$$\bar{R} = \frac{1}{m} \sum_i R_i$$

Note that

$$ER = d_2 \sigma \quad \text{and} \quad \operatorname{Var} R = d_3^2 \sigma^2$$

(see Chapter 4).

The constants c_4 and d_2, d_3 are tabulated in Appendix Table VI, p 702.

(Trial) control limits for the mean chart:

$$\begin{aligned}
\text{UCL} &= \bar{\bar{x}} + A_2\bar{R} \\
\text{CL} &= \bar{\bar{x}} \\
\text{LCL} &= \bar{\bar{x}} - A_2\bar{R}
\end{aligned}$$

Development.

$$ER = d_2\sigma$$

$$\Rightarrow \quad \hat{\sigma} = \frac{\bar{R}}{d_2} \quad \text{is unbiased for } \sigma$$

$$\Rightarrow \quad \hat{\sigma}/\sqrt{n} = \frac{\bar{R}}{d_2\sqrt{n}} \quad \text{estimates s.e.} = \sigma/\sqrt{n} \text{ without bias}$$

$$\Rightarrow \quad A_2 = \frac{3}{d_2\sqrt{n}}$$

The constant A_2 is tabulated in Appendix Table VI, p 702.

(Trial) control limits for the range chart:

$$\begin{aligned} \text{UCL} &= D_4 \bar{R} \\ \text{CL} &= \bar{R} \\ \text{LCL} &= D_3 \bar{R} \end{aligned}$$

Development.

$$\begin{aligned} & \sigma_R = \sqrt{\operatorname{Var} R} = d_3 \sigma \\ \Rightarrow \quad & \hat{\sigma}_R = d_3 \hat{\sigma} = d_3 \frac{\bar{R}}{d_2} \\ \Rightarrow \quad & \text{UCL} = \bar{R} + 3 \frac{d_3}{d_2} \bar{R} \\ \& \quad & \text{LCL} = \bar{R} - 3 \frac{d_3}{d_2} \bar{R} \end{aligned}$$

Thus,

$$\begin{aligned} D_4 &= 1 + \frac{3 d_3}{d_2} \\ D_3 &= 1 - \frac{3 d_3}{d_2} \end{aligned}$$

The constants D_3, D_4 are tabulated in Appendix Table VI, p 702.

In Example 6.1, there are $m = 25$ preliminary samples, each of size $n = 5$, and

$$\sum_i \bar{x}_i = 37.6400 \quad \Rightarrow \quad \bar{\bar{x}} = 1.5056$$

$$\sum_i R_i = 8.1302 \quad \Rightarrow \quad \bar{R} = 0.32521$$

The control values for the range chart are

$$\begin{aligned} \mathrm{UCL} &= D_4\bar{R} = 2.114 \cdot 0.32521 = 0.68749 \\ \mathrm{CL} &= 0.32521 \\ \mathrm{LCL} &= D_3\bar{R} = 0 \cdot 0.32521 = 0 \end{aligned}$$

One may compare using the range control chart and two-sided testing of

$$H_0 : \sigma = d_2\bar{R} = 2.326 \cdot 0.32521 = 0.7564$$

at level $\alpha = 0.0027$. Note that the two-sided test rejects if

$$\frac{4s^2}{0.7564^2} \leq \chi^2_{1-\alpha/2,n-1} = 0.1058 \quad \Leftrightarrow \quad s \leq 0.1230$$

or

$$\frac{4s^2}{0.7564^2} \geq \chi^2_{\alpha/2,n-1} = 17.8006 \quad \Leftrightarrow \quad s \geq 1.5957$$

for samples of size $n = 5$.

The control values for the mean chart are

$$\begin{aligned} \text{UCL} &= \bar{\bar{x}} + A_2\bar{R} = 1.5056 + 0.577 \cdot 0.32521 = 1.69325 \\ \text{CL} &= \bar{\bar{x}} = 1.5056 \\ \text{LCL} &= \bar{\bar{x}} - A_2\bar{R} = 1.5056 - 0.577 \cdot 0.32521 = 1.31795 \end{aligned}$$

One may compare using the mean control chart and the testing of

$$H_0 : \mu = 1.5056 \quad \text{vs} \quad H_1 : \mu \neq 1.5056$$

which rejects at level $\alpha = 0.0027$, if

$$\left| \frac{\bar{x} - 1.5056}{s/\sqrt{n}} \right| \geq t_{\alpha/2,n-1} = 6.6202$$

for samples of size $n = 5$.

An alternative could be to use a standardized control chart, where the points are plotted in standard units. The variable plotted on such a chart would be

$$t = \frac{\bar{x} - 1.5056}{s/\sqrt{n}}$$

with UCL $= 6.6202$ and LCL $= -6.6202$.

Sometimes the charts are based on standard values.

E.g, if σ is known, then, for the range chart,

$$\begin{aligned} \mathrm{UCL} &= D_2\sigma = (d_2 + 3d_3)\sigma \\ \mathrm{CL} &= \sigma \\ \mathrm{LCL} &= D_1\sigma = (d_2 - 3d_3)\sigma \end{aligned}$$

The constants D_1, D_2 are tabulated in Appendix Table VI, p 702.

Analogously for the mean chart.

Estimating process capability.

Let x denote a typical process output.

The fraction nonconforming items produced is

$$\begin{aligned} p &= P(x < \text{LSL or } x > \text{USL}) \\ &= P(x < \text{LSL}) + P(x > \text{USL}) \\ &= 2P(x < \text{LSL}) \end{aligned}$$

where the latter equality pressumes

$$\mu = \frac{\text{LSL} + \text{USL}}{2}$$

The process capability ratio (PCR), is

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma}$$

The percentage of the specification band that the process uses up, is

$$P = \left(\frac{1}{C_p}\right) 100\%$$

For instance, if the process capability estimate is

$$\hat{C}_p = 1.25$$

then the process uses up approximately 80% of the specification band.

Look at Figure 6.3.

The natural tolerance limits of the process are customary defined as 3σ above (UNTL) and below (LNTL) the process mean.

Note that there is no mathematical or statistical relationship between the control limits and the specification limits.

The OC curve for a mean control chart. The in control parameters are μ_0 and σ_0, where the latter is assumed known and constant. The β-risk or the probability of not detecting a shift in the mean of k standard deviations, is

$$\beta = P(\mathrm{LCL} \leq \bar{x} \leq \mathrm{UCL} | \mu_1 = \mu_0 + k\sigma_0)$$
$$= P(\bar{x} \leq \mathrm{UCL} | \mu_1 = \mu_0 + k\sigma_0) - P(\bar{x} \leq \mathrm{LCL} | \mu_1 = \mu_0 + k\sigma_0)$$
$$= \Phi(L - k\sqrt{n}) - \Phi(-L - k\sqrt{n})$$

where $L = 3$, typically.

This is an even function of k. See Figure 6.13.

The in control run length is

$$\mathrm{ARL}_0 = \frac{1}{\alpha}$$

The out-of-control run length is

$$\mathrm{ARL}_1 = \frac{1}{1 - \beta_1}$$

Construction and operation of $\bar{x}$ and s charts.

When standard values are given for μ and σ, the mean chart has three-sigma limits

$$\mathrm{UCL} = \mu + 3\sigma/\sqrt{n}$$
$$\mathrm{LCL} = \mu - 3\sigma/\sqrt{n}$$

The three-sigma control limits for the s chart is

$$\mathrm{UCL} = (c_4 + 3\sqrt{1 - c_4^2})\sigma = B_6\sigma$$
$$\mathrm{LCL} = (c_4 - 3\sqrt{1 - c_4^2})\sigma = B_5\sigma$$

since

$$Es = c_4\sigma \quad \text{and} \quad \mathrm{Var}\, s = (1 - c_4^2)\sigma^2$$

Note also,

$$\mathrm{CL} = c_4\sigma$$

When no standard is given, we begin with m preliminary samples, each of size n. The average of m standard deviations is

$$\bar{s} = \frac{1}{m} \sum_i s_i$$

The statistic $\bar{s}/c_4$ is unbiased for s. Hence, by comparing with the case when standard values are given,

$$\begin{aligned} \mathrm{UCL} &= B_6 \frac{\bar{s}}{c_4} = \bar{s} + 3\frac{\sqrt{1-c_4^2}}{c_4}\bar{s} = B_4\bar{s} \\ \mathrm{CL} &= \bar{s} \\ \mathrm{LCL} &= B_5 \frac{\bar{s}}{c_4} = \bar{s} - 3\frac{\sqrt{1-c_4^2}}{c_4}\bar{s} = B_3\bar{s} \end{aligned}$$

Note that

$$B_4 = B_6/c_4 \quad \text{and} \quad B_3 = B_5/c_4$$

The associated mean control chart has

$$\begin{aligned} \mathrm{UCL} &= \bar{\bar{x}} + \frac{3\bar{s}}{c_4\sqrt{n}} = \bar{\bar{x}} + A_3\bar{s} \\ \mathrm{CL} &= \bar{\bar{x}} \\ \mathrm{LCL} &= \bar{\bar{x}} - \frac{3\bar{s}}{c_4\sqrt{n}} = \bar{\bar{x}} - A_3\bar{s} \end{aligned}$$

Reading hints.

The remaining part of Section 6.3 Control Charts for $\bar{x}$ and s is left for self-study.

Section 6.4 The Shewhart Control Chart for Individual measurements (the case $n = 1$) may be omitted.

Section 6.5 Summary of Procedures for $\bar{x}$, s and R Charts, and Section 6.6 Applications of Variables Control Charts are also left for self-study.

4c) Attributes Control Charts

Control chart for fraction nonconforming.

If D is the number of units of product that are nonconforming, then

$$D \sim \mathrm{Bin}(n, p)$$

where n is the sample size, and p is the true proportion nonconforming units.

The sample fraction nonconforming is

$$\hat{p} = \frac{D}{n}$$

Note,

$$E\hat{p} = p \quad \text{and} \quad \mathrm{Var}\,\hat{p} = \frac{p(1-p)}{n}$$

If standard is given,

$$\begin{aligned} \mathrm{UCL} &= p + 3\sqrt{\frac{p(1-p)}{n}} \\ \mathrm{CL} &= p \\ \mathrm{LCL} &= p - 3\sqrt{\frac{p(1-p)}{n}} \end{aligned}$$

If no standard is given, replace p with

$$\bar{p} = \frac{1}{m}\sum_i \hat{p}_i$$

where the number of phase I samples $m \gtrapprox 20$ and $\hat{p}_i$ is the ith sample fraction nonconforming.

The so obtained control limits should be regarded as trial limits.

Consider Example 7.1.

Design of the fraction nonconforming control chart.

Three parameters must be specified: the sample size n, the sample frequency and the width of the control limits.

Suppose $p = 0.01$. If $n = 8$ and $L = 3$, then

$$\text{UCL} = 0.01 + 3\sqrt{\frac{0.01 \cdot 0.99}{8}} = 0.1155$$

If there is one nonconforming unit in the sample,

$$\hat{p} = \frac{1}{8} = 0.125 > \text{UCL}$$

Montgomery says that in many cases it is unreasonable to conclude that the process is out of control on observing a single nonconforming item.

To avoid this pitfall, the sample size must be larger than 8. If we want the probability of at least one nonconforming item in the sample to be around 0.95, then $n \approx 300$. Note that, even for such a big sample size, $np \approx 3$ and the c.l.t is not applicable.

Suppose we want to detect a shift from $p = 0.01$ to

$$p_1 = 0.05 \text{ to be } 1 - \beta_1 = 0.50$$

Then

$$0.5 = P(\hat{p} > \text{UCL}|p_1)$$

$$= P\left(\frac{\hat{p} - p_1}{\sqrt{\frac{p_1(1-p_1)}{n}}} > \frac{p - p_1 + L\sqrt{\frac{p(1-p)}{n}}}{\sqrt{\frac{p_1(1-p_1)}{n}}}\right)$$

Assuming the distribution of $\hat{p}|p_1$ symmetric around its mean p_1 (this is weaker than assuming the c.l.t be applicable, which not necessarily is the case),

$$p_1 - p = L\sqrt{\frac{p(1-p)}{n}}$$

$$\Rightarrow \; p_1 = p + L\sqrt{\frac{p(1-p)}{n}}$$

That is,

$$\text{UCL} = p_1$$

and

$$n = \left(\frac{L}{p_1 - p}\right)^2 p(1-p) \approx 56$$

Another useful criterion is to choose n large enough so that the control chart will have LCL > 0. That is,

$$p > L\sqrt{\frac{p(1-p)}{n}}$$

or

$$n > \frac{1-p}{p}\,L^2 \approx 171$$

Reading hints.

Section 7.2.2 Variable Sample Size is left for self-study.

Section 7.2.3 is omitted.

The OC curve.

The probability of a type II error for the fraction nonconforming chart, may be computed from

$$\begin{aligned}\beta &= P(\mathrm{LCL} < \hat{p} < \mathrm{UCL}|p)\\ &= P(\hat{p} < \mathrm{UCL}|p) - P(\hat{p} \le \mathrm{LCL}|p)\\ &= P(D < n\mathrm{UCL}|p) - P(D \le n\mathrm{LCL}|p)\end{aligned}$$

If LCL < 0, omit the right-most term.

The ARL calculations are not different from the variables case.

Reading hints.

Section 7.3 Control Charts for Nonconformities (Defects) is omitted.

Section 7.4 Choice between Attributes and Variables Control Charts is left for self-study. So is also Section 7.5 Guidelines for Implementing Control Charts.