Chapter 7

(7) Ignoring the finite population correction (makes sense when dealing with big population, like in this case), the formula for estimating standard error of a proportion is: $\hat{S}_p = \sqrt{\frac{p(1-p)}{n}}$

According to our information $\hat{S}_p = 0.02$ and $p = 0.15$, then

$$0.02 = \sqrt{\frac{0.15(1-0.15)}{n}} \Rightarrow n = 318.75 \approx 319$$

(15) a) $P(|\bar{X}-\mu| > 200) = P(-(\bar{X}-\mu) < 200) + P((\bar{X}-\mu) > 200)$

$= 2P((\bar{X}-\mu) > 200) = 2[1 - P((\bar{X}-\mu) < 200)] = 2\left[1 - P\left(\frac{\bar{X}-\mu}{\sigma_{\bar{x}}} < \frac{200}{\sigma_x}\right)\right]$

$= 2\left[1 - \Phi\left(\frac{200}{\sigma_x}\right)\right]$ where $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}\sqrt{1 - \frac{n-1}{N-1}}$

$20 \le n \le 100$, $N = 393$ and $\sigma = \frac{1}{N}\sum_{i=1}^{N} x_i^2 - \mu^2 = 589.$

b) $P(|\bar{X}-\mu| > \Delta) = 2P(\bar{X}-\mu > \Delta) = 2\left[1 - P\left(\frac{\bar{X}-\mu}{\sigma_{\bar{x}}} < \frac{\Delta}{\sigma_x}\right)\right]$

$= 2\left[1 - \Phi\left(\frac{\Delta}{\sigma_{\bar{x}}}\right)\right]$

$P(|\bar{X}-\mu| > \Delta) = 0.10 \iff 2\left[1 - \Phi\left(\frac{\Delta}{\sigma_{\bar{x}}}\right)\right] = 0.10$

$\iff 1 - \Phi\left(\frac{\Delta}{\sigma_{\bar{x}}}\right) = 0.05 \iff \Phi\left(\frac{\Delta}{\sigma_{\bar{x}}}\right) = 0.95 \iff \frac{\Delta}{\sigma_{\bar{x}}} = 1.645$

n	$\sigma_{\bar{x}}$	Δ_1	Δ_2
20	128.626	211.6	86.8
40	88.48	145.6	59.7
80	58.914	96.9	39.3

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}\sqrt{1 - \frac{n-1}{N-1}} = \frac{589.7}{\sqrt{20}}\sqrt{1 - \frac{19}{392}}$

$= 128.626$

$\Rightarrow \Delta = 1.645 * 128.626 = 211.590$

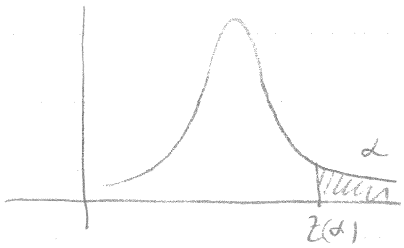
$$P(|\bar{X}-\mu| > \Delta) = 0.50 \iff 2\left[1-\phi\left(\frac{\Delta}{\sigma_{\bar{x}}}\right)\right] = 0.50$$

$$\iff 1-\phi\left(\frac{\Delta}{\sigma_{\bar{x}}}\right) = 0.25 \iff \phi\left(\frac{\Delta}{\sigma_{\bar{x}}}\right) = 0.75$$

$$\iff \frac{\Delta}{\sigma_{\bar{x}}} = 0.675 \iff \Delta = 0.675 * 128.626 = 86.823$$

17 = No, it means that the probability of the population mean lying within this interval is 0.9, which doesn't imply that 90 % of the population lies within this interval.

- The CI is computed from $P\left(-z(\alpha/2) \leq \frac{\bar{X}-\mu}{\sigma_{\bar{x}}} \leq z(\alpha/2)\right) = 1-\alpha$ where $z(\alpha)$ is the number such that the area under $N(0,1)$ to the right of $z(\alpha)$ is α



≡ Also: For 100 CI with this confidence level, 95 will contain μ

21 The width of the 95% CI for the mean is $k s_{\bar{x}} = k \frac{\hat{\sigma}}{\sqrt{n}}$ ($k = 1.96$). To halve it equal thus to

$$\frac{k s_{\bar{x}}}{2} = \frac{k \hat{\sigma}}{2\sqrt{n}} = \frac{k \hat{\sigma}}{\sqrt{4n}}$$ therefore the sample size should be

multiplied by 4

(31) The 95% CI is formed as $T \pm 1.96 S_T$, so its width is $3.92 S_T$ and we want $3.92 S_T = 500 \iff S_T = 127.551 \iff$

$N S_{\hat{p}} = 127.551 \iff S_{\hat{p}} = \frac{127.551}{8000} = 0.0159$ and $S_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n-1}}\sqrt{1-\frac{n}{N}}$

$0.0159 = \sqrt{\frac{\hat{p}(1-\hat{p})}{n-1}}\sqrt{1-\frac{n}{N}}$ $\quad \hat{p} = 0.12 \Rightarrow 0.0159 = \sqrt{\frac{0.12(1-0.12)}{n-1}}\sqrt{1-\frac{n}{8000}}$

$$0.0159^2 = \frac{0.1056}{n-1}\left(1-\frac{n}{8000}\right)$$

$$= \frac{0.1056}{n-1} - \frac{0.1056\,n}{8000(n-1)}$$

$$= \frac{844.8 - 0.1056\,n}{8000(n-1)}$$

$2.0225(n-1) = 844.8 - 0.1056n$

$2.1281n = 846.8225$

$n = 397.9242$

$n \approx 398$

(33) We know that $p_1 = p_2 = 0.5$ and we want $S_D \leq 0.02$, so let

$\sqrt{\frac{p_1(1-p_1)}{n} + \frac{p_2(1-p_2)}{n}} = 0.02 \iff \sqrt{\frac{0.5 \cdot 0.5}{n} + \frac{0.5 \cdot 0.5}{n}} = 0.02 \iff$

$\sqrt{\frac{0.5}{n}} = 0.02 \iff n = 1250$

(48) a) $R = \frac{\bar{Y}}{\bar{X}} = \frac{\frac{1}{n}\sum Y_i}{\frac{1}{n}\sum X_i} = \frac{\sum Y_i}{\sum X_i} = \frac{10000}{320} = 31.25$

$\bar{X} = 3.2$

$\bar{Y} = 100$

b) $S_x^2 \overset{*}{=} \frac{1}{n-1}\left[\sum X_i^2 - \frac{(\sum X_i)^2}{n}\right]$

$= \frac{1}{99}\left[1250 - \frac{(320)^2}{100}\right] = 2.2828$

$S_Y^2 = \frac{1}{99}\left[1100000 - \frac{(10\,000)^2}{100}\right] = 1010.101$

$S_{XY} = \frac{1}{n-1}\left[\sum X_i Y_i - n\bar{X}\bar{Y}\right]$

$= \frac{1}{n-1}\left[\sum X_i Y_i - n\frac{1}{n}\sum X_i \frac{1}{n}\sum Y_i\right]$

$= \frac{1}{n-1}\left[\sum X_i Y_i - \frac{\sum X_i \cdot \sum Y_i}{n}\right]$

$= \frac{1}{99}\left[36\,000 - \frac{320 \cdot 10\,000}{100}\right] = 40.4040$

$S_R^2 = \frac{1}{n}\left(1 - \frac{n-1}{N-1}\right)\frac{1}{\bar{X}^2}\left(R^2 S_x^2 + S_Y^2 - 2RS_{XY}\right)$

$= \frac{1}{99}\left(1 - \frac{99}{99\,999}\right)\frac{1}{(3.2)^2}\left(31.25^2 \cdot 2.2828 + 1010.1 - 2 \cdot 31.25 \cdot 40.4040\right)$

$= 0.839$

90 % CI: $R \pm z_{\alpha/2} S_R \Leftrightarrow 31.25 \pm 1.96 \cdot 0.839 \Leftrightarrow 31.25 \pm 1.6444$

c) $T = N\bar{Y} = 100000 \cdot 100 = 10^7$

90% CI : $T \pm z_{\alpha/2} S_T \iff T \pm z_{\alpha/2} N S_{\bar{Y}}$

$S_{\bar{Y}} = \frac{S_Y}{\sqrt{n}} \sqrt{1 - \frac{n}{N}} = \frac{\sqrt{1010.101}}{\sqrt{100}} \sqrt{1 - \frac{100}{100000}} = 3.1766$

$10^7 \pm 1.645 \cdot 100000 \cdot 3.1766 \iff 10^7 \pm 522550$

Note: $S_Y = 31.7821 \Rightarrow 10^7 \pm 5228153$

$$* \quad S_x^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2 = \frac{1}{n-1} \sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2)$$

$$= \frac{1}{n-1} \left[\sum x_i^2 - 2\bar{x} \sum x_i + \sum \bar{x}^2 \right]$$

$$= \frac{1}{n-1} \left[\sum x_i^2 - 2\bar{x}\, n \left(\frac{\sum x_i}{n} \right) + n\bar{x}^2 \right]$$

$$= \frac{1}{n-1} \left[\sum x_i^2 - 2n\bar{x}^2 + n \left(\frac{\sum x_i}{n} \right)^2 \right]$$

$$= \frac{1}{n-1} \left[\sum x_i^2 - \frac{(\sum x_i)^2}{n} \right]$$

Chapter 6

⑤ Show that if $X \sim F_{n,m} \Rightarrow X^{-1} \sim F_{m,n}$

Let $Y = X^{-1}$, then

$$F_Y(y) = P(Y \le y) = P(X^{-1} \le y) = P\left(X \ge \frac{1}{y}\right) = 1 - P(X \le 1/y) = 1 - F_X(1/y)$$

$$f_Y(y) = F_Y'(y) = \frac{d}{dy}\left[1 - F_X(1/y)\right] = -f_X(1/y)\left(-\frac{1}{y^2}\right) = \frac{1}{y^2} f_X(1/y)$$

$$f_X(x) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{n}{m}\right)^{\frac{n}{2}} x^{\frac{n}{2}-1}\left(1 + \frac{n}{m}x\right)^{-\frac{n+m}{2}}$$

$$\frac{1}{y^2} f_X(1/y) = \frac{1}{y^2}\frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{n}{m}\right)^{\frac{n}{2}}\left(\frac{1}{y}\right)^{\frac{n}{2}-1}\left(1 + \frac{n}{m}\frac{1}{y}\right)^{-\frac{n+m}{2}} =$$

$$\frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\frac{m}{n}\right)^{\frac{m}{2}}\left(\frac{m}{n}\right)^{-\frac{m}{2}}\left(\frac{n}{m}\right)^{\frac{n}{2}}\left(\frac{1}{y}\right)^{\frac{n}{2}+1}\left(1 + \frac{n}{m}\frac{1}{y}\right)^{-\frac{m+n}{2}} =$$

$$\frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\frac{m}{n}\right)^{\frac{m}{2}}\left(\frac{m}{n}\right)^{-\frac{m}{2}}\left(\frac{m}{n}\right)^{-\frac{n}{2}} y^{-\frac{n}{2}-1}\left(1 + \frac{n}{m}\frac{1}{y}\right)^{-\frac{m+n}{2}} =$$

$$\frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\frac{m}{n}\right)^{\frac{m}{2}}\left(\frac{m}{n}\right)^{-\frac{m+n}{2}} y^{\frac{m}{2}-1} y^{-\frac{m}{2}+1} y^{-\frac{n}{2}-1}\left(1 + \frac{n}{m}\frac{1}{y}\right)^{-\frac{m+n}{2}} =$$

$$\frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\frac{m}{n}\right)^{\frac{m}{2}} y^{\frac{m}{2}-1}\left(\frac{m}{n}\right)^{-\frac{m+n}{2}} y^{-\frac{m+n}{2}}\left(1 + \frac{n}{m}\frac{1}{y}\right)^{-\frac{m+n}{2}} =$$

$$\frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\frac{m}{n}\right)^{\frac{m}{2}} y^{\frac{m}{2}-1}\left[\frac{m}{n}y\left(1 + \frac{n}{m}\frac{1}{y}\right)\right]^{-\frac{m+n}{2}} =$$

$$\frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\frac{m}{n}\right)^{\frac{m}{2}} y^{\frac{m}{2}-1}\left(1 + \frac{m}{n}y\right)^{-\frac{m+n}{2}}$$

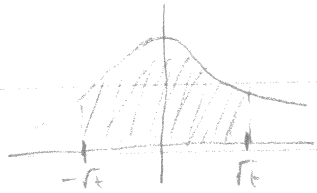
$\Rightarrow Y \sim F_{m,n}$

- Short version. Definition: If $U \sim \chi^2_{(n)}$ and $V \sim \chi^2_{(m)}$ are independent then $X = \frac{U/n}{V/m} \sim F_{n,m}$. In this case $X^{-1} = \frac{V/m}{U/n} \sim F_{m,n}$

- Another version. General formula for quotient transformations

$$Z = \frac{Y}{X} \Rightarrow f_Z(z) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xz)\, dx$$

⑥ Show that if $T \sim t_n$ then $T^2 \sim F_{1,n}$



$$F_{T^2}(t) = P(T^2 \le t) = P(|T| \le t^{\frac{1}{2}}) = P(-t^{\frac{1}{2}} \le T \le t^{\frac{1}{2}}) = 1 - 2P(T \le -\sqrt{t})$$

$$f_{T^2}(t) = F'_{T^2}(t) = -2F'(-\sqrt{t})\left(\frac{-1}{2\sqrt{t}}\right) = \frac{1}{\sqrt{t}} f_T(-\sqrt{t})$$

$$f_T(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}}$$

$$t^{-\frac{1}{2}} f_T(-t^{\frac{1}{2}}) = t^{-\frac{1}{2}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left[1 + \frac{1}{n}(-t^{\frac{1}{2}})^2\right]^{-\frac{n+1}{2}} =$$

$$\frac{\Gamma\left(\frac{1+n}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\frac{1}{n}\right)^{\frac{1}{2}} t^{\frac{1}{2}-1}\left(1 + \frac{1}{n}t\right)^{-\frac{1+n}{2}}$$

Note: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

- Short version. Definition: If $Z \sim N(0,1)$ and $U \sim \chi^2_n$ are independent ~~then~~ $T = \frac{Z}{\sqrt{U/n}} \sim t_{(n)}$. In this case $T^2 = \frac{Z^2}{U/n} = \frac{V/1}{U/n}$ where $V \sim \chi^2_{(1)}$ then $T^2 \sim F_{1,n}$

Chapter 8

①

n	Observed
0	5267
1	4436
2	1800
3	534
4	111
5+	21
	12 169

We first estimate the parameter λ of the Poisson distribution as the observed mean emission rate (total number of emission divided by total time) $\Rightarrow \hat{\lambda} = 0.8392$

$$\pi_n = P(X=n) = \frac{\lambda^n e^{-\lambda}}{n!}$$ p_n: probability of falling in cell n

$$p_5 = P(X \geq 5) = 1 - P(X < 4) = 1 - (\pi_1 + \pi_2 + \pi_3 + \pi_4)$$

p_n	Expected
0.4321	5258
0.3626	4413
0.1521	1851
0.0426	518
0.0089	108
0.0017	21

Under the assumption $X_1, X_2, \dots X_{12169}$ are independent Poisson random variables, the number of observations falling in a given cell follows a binomial distribution with mean $12169 p_n$

So they qualitatively match the observed counts

⑤ X has distribution $P(X=1) = \theta$, $P(X=2) = 1-\theta$

Sample: $x_1 = 1$, $x_2 = 2$, $x_3 = 2$

a) Find the method of moments estimate of θ

$$E(X) = \sum_i x P(X=x) = 1\theta + 2(1-\theta) = \theta + 2 - 2\theta = 2 - \theta$$

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{1}{3}(1+2+2) = 5/3$$

Method of moments: $E(X) = \hat{\mu} \Rightarrow 2 - \theta = 5/3 \Rightarrow \hat{\theta} = 2 - 5/3 = 1/3$

b) Likelihood function: $\text{lik}(\theta) = \prod_{i=1}^{n} f(X_i | \theta) = \theta (1-\theta)^2$

c) MLE of θ $\quad l(\theta) = \log(\theta) + 2\log(1-\theta)$

*

$l'(\theta) = \frac{1}{\theta} - \frac{2}{1-\theta}$

$l'(\theta) = 0 \iff \frac{1}{\theta} - \frac{2}{1-\theta} = 0$

$\frac{1}{\theta} = \frac{2}{1-\theta}$

$1-\theta = 2\theta$

$\hat{\theta} = 1/3$

d) Posterior density $\propto$ Likelihood X Prior density

$f_{\Theta|X}(\theta|x) \propto \theta(1-\theta)^2 \, 1 = \theta^{2-1}(1-\theta)^{3-1}$

$\Rightarrow f_{\Theta|X}(\theta|x) = \frac{1}{B(2,3)} \theta(1-\theta)^2 \rightarrow$ Beta(2,3)

$E(\Theta) = \frac{\alpha}{\alpha+\beta} = \frac{2}{2+3} = \frac{2}{5}$

(7) $X \sim Geo(p) \Rightarrow f(x) = p(1-p)^{x-1}$ Assume an iid sample of size n

a) Find the method of moments estimate of p

$E(X) = \frac{1}{p}$ $\quad \hat{\mu} = \bar{X}$

MOM: $\frac{1}{p} = \bar{X} \Rightarrow p = 1/\bar{X}$

b) Find the MLE of p

$lik(\theta) = \prod_{i=1}^{n} f(x_i|p) = \prod_{i=1}^{n} p(1-p)^{x_i-1} = p^n(1-p)^{\Sigma x_i - n}$

$l(\theta) = n\log(p) + (\Sigma x_i - n)\log(1-p)$

$l'(\theta) = \frac{n}{p} - \frac{\Sigma x_i - n}{1-p}$ $\quad l'(\theta) = 0 \iff \frac{n}{p} - \frac{\Sigma x_i - n}{1-p} = 0 \iff$

$\frac{n}{p} = \frac{\Sigma x_i - n}{1-p} \iff n(1-p) = (\Sigma x_i - n)p \iff n = \Sigma x_i p \iff$

$\hat{p} = \frac{n}{\Sigma x_i} = \frac{1}{\bar{x}}$

*

c) The large sample distribution of the MLE is approximately $N(\theta_0, 1/nI(\theta_0))$ where $I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} \log P(X|\theta)\right]$ and θ_0 is the true value of θ. Thus, the asymptotic variance is (here $\theta_0 = p$)

$$\frac{1}{nI(\theta_0)} = -\frac{1}{-nE\left[\frac{\partial^2}{\partial p^2}\{p(1-p)^{x-1}\}\right]} = -\frac{1}{nE\left[-\frac{1}{p^2} - \frac{x-1}{(1-p)^2}\right]} =$$

$$\frac{1}{\frac{n}{p^2} + \frac{n}{(1-p)^2}\left[\left(\frac{1}{p}\right) - 1\right]} = \frac{1}{\frac{n}{p^2} + \frac{n}{p(1-p)^2} - \frac{n}{(1-p)^2}} =$$

$$\frac{1}{\frac{n(1-p)^2 + np - np^2}{p^2(1-p)^2}} = \frac{p^2(1-p)^2}{n+2np+np^2+np-np^2} = \frac{p^2(1-p)^2}{n-np} =$$

$$\frac{p^2(1-p)^2}{n(1-p)} = \frac{p^2(1-p)}{n}$$

d) Posterior $\propto$ Likelihood $\cdot$ Prior

$$f_{P|X}(p,x) \propto p^n(1-p)^{\sum x_i - n} \cdot 1 \Rightarrow f_{P|X}(p,x) = \frac{p^{(n+1)-1}(1-p)^{(\sum x_i - n+1)-1}}{B(n+1, \sum x_i - n+1)}$$

$$E(P) = \frac{\alpha}{\alpha+\beta} = \frac{n+1}{n+1+\sum x_i - n+1} = \frac{n+1}{\sum x_i + 1}$$

(19) Suppose $X_1, X_2, \dots, X_n$ are iid $N(\mu, \sigma^2)$

a) If μ is known, what is the MLE of σ?

From the log likelihood function we obtain the following partial derivatives:

$$\frac{\partial l}{\partial \mu} = \frac{1}{\sigma^2}\sum_i (X_i - \mu) \qquad \frac{\partial l}{\partial \sigma} = -\frac{n}{\sigma} + \sigma^{-3}\sum_i (X_i - \mu)^2$$

Setting the second one equal to zero we obtain: $0 = -\frac{n}{\sigma} + \sigma^{-3}\sum(X_i-\mu)^2 \Leftrightarrow$
$-n = \sigma^{-2}\sum(X_i-\mu)^2 \Leftrightarrow \sigma^2 = \frac{\sum(X_i-\mu)^2}{n} \Leftrightarrow \hat{\sigma} = \sqrt{\frac{1}{n}\sum(X_i-\mu)^2}$

*

b) If σ is know what is the MLE for μ?

Now setting the first partial derivative to zero we obtain

$$0 = \frac{1}{\sigma^2}\sum(X_i - \mu) \iff 0 = \sum X_i - n\mu \iff \hat{\mu} = \frac{1}{n}\sum X_i = \bar{X}$$

c) In the case σ known, does any other unbiased estimate of μ have smaller variance?

According to the Cramér-Rao inequality, for an estimate $\hat{\mu}$ of μ

$$Var(\hat{\mu}) \geq \frac{1}{nI(\mu)}$$

$$\log f(X|\mu) = -\log\sigma - \frac{1}{2}\log(2\pi) - \frac{1}{2\sigma^2}(X-\mu)^2$$

$$\frac{\partial}{\partial\mu}\log f(X|\mu) = -\frac{1}{2\sigma^2}\,2\,(X-\mu)(-1) = \frac{1}{\sigma^2}(X-\mu)$$

$$I(\mu) = E\left\{\left[\frac{\partial}{\partial\mu}\log f(X|\mu)\right]^2\right\} = E\left\{\left[\frac{1}{\sigma^2}(X-\mu)\right]^2\right\} = \frac{1}{\sigma^4}E\{(X-\mu)^2\} = \frac{\sigma^2}{\sigma^4} = \frac{1}{\sigma^2}$$

Thus $Var(\hat{\mu}) \geq \dfrac{1}{n\,1/\sigma^2} = \dfrac{\sigma^2}{n}$

But $Var(\hat{\mu}) = Var(\bar{X}) = Var\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2}\sum Var(X_i) = \frac{1}{n^2}n\sigma^2 = \frac{\sigma^2}{n}$

So $\hat{\mu}$ reaches the Cramér-Rao lower bound, therefore there's no other estimate with smaller variance