Bayesian statistics
Chapter 8

(5) X discrete random variable with $P(X=1)=\theta$ and $P(X=2)=1-\theta$

Data: $x_1=1$, $x_2=2$, $x_3=2$

$\hat{\theta}_{MOM}=\frac{1}{3}$ $\hat{\theta}_{MLE}=\frac{1}{3}$

d) If $\Theta \sim U[0,1]$ what's the posterior density?

Posterior density $\propto$ Likelihood $\cdot$ Prior density

likelihood function: $L(\theta)=\prod_{i=1}^{n} f(x_i|\theta)=\theta(1-\theta)^2$

$$f_{\Theta|X}(\theta|x) \propto \theta(1-\theta)^2 \cdot 1 = \theta^{2-1}(1-\theta)^{3-1} \Rightarrow f_{\Theta|X}(\Theta|X)=\frac{1}{B(2,3)}\theta(1-\theta)^2$$

or For $X \sim Bin(3,p)$ and $\Theta \sim U[0,1]=B(1,1)$
the posterior distribution is $B(1+1,1+3-1)=B(2,3)$ ($x=$ # successes)

$$\hat{\theta}_{PME}=E(\Theta|X=x)=\frac{2}{2+3}=\frac{2}{5}$$

Normal conjugates

Data sample of size n: $X_i \sim N(\mu,\sigma^2)$ $i=1,\dots,n$

Prior for $\mu \sim N(\mu_0,\sigma_0^2)$ (variance known). $\mu=\Theta$ random variable
$\Rightarrow \Theta \sim N(\theta_0,\sigma_0^2)$

$$f_{\Theta|X}(\theta|x) \propto f_{X|\Theta}(x|\theta)\cdot f_\Theta(\theta)$$

$$=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2}\frac{(x_i-\theta)^2}{\sigma^2}\right\}\cdot\frac{1}{\sqrt{2\pi\sigma_0^2}}\exp\left\{-\frac{1}{2}\frac{(\theta-\theta_0)^2}{\sigma_0^2}\right\}$$

$$=\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}}\left(\frac{1}{2\pi\sigma_0}\right)^{\frac{1}{2}}\prod_{i=1}^{n}\exp\left\{-\frac{1}{2}\frac{(x_i-\theta)^2}{\sigma^2}\right\}\exp\left\{-\frac{1}{2}\frac{(\theta-\theta_0)^2}{\sigma_0^2}\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2+\frac{1}{\sigma_0^2}(\theta-\theta_0)^2\right]\right\}$$

$$\left\ulcorner \sum(x_i-\theta)^2 = \sum(x_i-\bar{x})^2 + n(\theta-\bar{x})^2 \right\lrcorner$$

$$= \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\theta-\bar{x})^2 + \frac{1}{\sigma_0^2}(\theta-\theta_0)^2\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{n}{\sigma^2}(\theta-\bar{x})^2 + \frac{1}{\sigma_0^2}(\theta-\theta_0)^2\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\frac{n}{\sigma^2}(\theta^2-2\theta\bar{x}+\bar{x}^2) + \frac{1}{\sigma_0^2}(\theta^2-2\theta\theta_0+\theta_0^2)\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}\right)\theta^2 - \left(\frac{n}{\sigma^2}\bar{x}+\frac{1}{\sigma_0^2}\theta_0\right)2\theta\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}\right)\left[\theta^2 - \left(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}\right)^{-1}\left(\frac{n}{\sigma^2}\bar{x}+\frac{1}{\sigma_0^2}\theta_0\right)2\theta\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}\right)\left[\theta^2 - 2\theta\left(\frac{n\sigma_0^2\bar{x}+\sigma^2\theta_0}{n\sigma_0^2+\sigma^2}\right)\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}\right)\left(\theta - \frac{n\sigma_0^2\bar{x}+\sigma^2\theta_0}{n\sigma_0^2+\sigma^2}\right)^2\right\} \quad \text{looks like} \quad \exp\left\{-\frac{1}{2}\sigma_p^{-1}(\theta-\theta_p)^2\right\}$$

$$\to N\left(\frac{n\sigma_0\bar{x}+\sigma^2\theta_0}{n\sigma_0^2+\sigma^2}, \left(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}\right)^{-1}\right) = N\left(\frac{\sigma^2}{n\sigma_0^2+\sigma^2}\theta_0 + \left(1-\frac{\sigma^2}{n\sigma_0^2+\sigma^2}\right)\bar{x}, \frac{\sigma^2}{n\sigma_0^2+\sigma^2}\sigma_0^2\right)$$

$$= N\left(c_n\theta_0 + (1-c_n)\bar{x}, c\sigma_0^2\right) \qquad c_n = \frac{\sigma^2}{n\sigma_0^2+\sigma^2} \quad \text{if } n=1 \quad c_1 = \frac{\sigma^2}{\sigma_0^2+\sigma^2}$$

If $\frac{1}{\sigma_0^2}$ is very small (large σ_0^2 and therefore a non-informative prior)

then $\Theta \sim N\left(\bar{x}, \frac{\sigma^2}{n}\right)$ as in MLE

Confidence interval for a normal population mean is given by
$\bar{X} \pm z(\alpha/2)\sigma_{\bar{x}}$
Credibility interval is constructed on the same fashion but following the posterior distribution (the Bayes estimator of the mean would be the mean value of the posterior distribution