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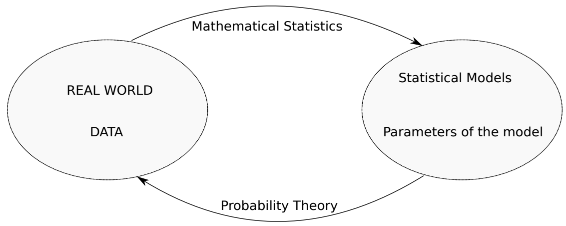

Chapter 7. Survey sampling

1 Random sampling

Population = set of elements $\{1, 2, \ldots, N\}$ labeled by values $\{x_1, x_2, \ldots, x_N\}$.
PD = population distribution of x-values.

> Pick at random one element from the population.
> Its x-value $X \sim$ PD is a random variable with the population distribution.

Types of x-values (data):
continuous, discrete, categorical, dichotomous (2 categories).
General population parameters
population mean $\mu = \mathrm{E}(X)$,
population standard deviation $\sigma = \sqrt{\mathrm{Var}(X)}$,
population proportion p (dichotomous data).
Two methods of studying PD and population parameters:
enumeration - expensive, sometimes impossible,
random sample: n random observations $(X_1, \ldots, X_n)$,
N is the population size, n is the sample size.

> *Randomisation* is a guard against investigator's biases even unconscious

Sampling with and without replacement
Sampling without replacement produces so called Simple Random Sample:
negative dependence $\mathrm{Cov}(X_i, X_j) = -\frac{\sigma^2}{N-1}$,
to prove use $X_1 + \ldots + X_N =$ const. and the addition rule of variance.
Sampling with replacement:
IID sample: Independent Identically Distributed observations,
easier to analyse,
good approximation of the simple random sample if n/N is small.

Example. Students heights: height in cm = discrete data, gender = dichotomous data.

2 Point estimates

Population parameter θ estimation uses a point estimate $\hat\theta = \hat\theta(X_1, \dots, X_n)$.
Sampling distribution of $\hat\theta$ around unknown θ: different values $\hat\theta$ observed for different samples.
Mean square error

$\mathrm{E}(\hat\theta - \theta)^2 = \left[\mathrm{E}(\hat\theta) - \theta\right]^2 + \sigma^2_{\hat\theta}$

$\mathrm{E}(\hat\theta) - \theta =$ systematic error, bias, lack of accuracy; $\sigma_{\hat\theta} =$ random error, lack of precision.

Desired properties of point estimates:

$\hat\theta$ is an unbiased estimate of θ, if $\mathrm{E}(\hat\theta) = \theta$,

$\hat\theta$ is consistent, if $\mathrm{E}(\hat\theta - \theta)^2 \to 0$ as $n \to \infty$.

Sample mean $\bar X = \frac{X_1+\ldots+X_n}{n}$ is an unbiased and consistent estimate of μ

$$\mathrm{Var}(\bar X) = \begin{cases} \sigma^2/n & \text{if IID sample} \\ \frac{\sigma^2}{n}(1 - \frac{n-1}{N-1}) & \text{if simple random sample} \end{cases}$$

Finite population correction $1 - \frac{n-1}{N-1}$ can be neglected if sample proportion $\frac{n}{N}$ is small.

Dichotomous data: $\mathrm{P}(X_i = 1) = p$, $\mathrm{P}(X_i = 0) = q$, $\mu = p$, $\sigma^2 = pq$, population proportion p. Sample proportion $\hat p = \bar X$ is an unbiased and consistent estimate of p.

Sample variance $s^2 = \frac{1}{n-1}\sum(X_i - \bar X)^2$, where s is the sample standard deviation.
Other formulae:

$s^2 = \frac{n}{n-1}(\overline{X^2} - \bar X^2)$, where $\overline{X^2} = \frac{1}{n}(X_1^2 + \ldots + X_n^2)$,

dichotomous data case $s^2 = \frac{n}{n-1}\hat p \hat q$.

Sample variance is an unbiased estimate of σ^2

$$\mathrm{E}(s^2) = \begin{cases} \sigma^2 & \text{if IID sample} \\ \sigma^2 \frac{N}{N-1} & \text{if simple random sample.} \end{cases}$$

Standard errors of $\bar X$ and $\hat p$ for simple random sample: $s_{\bar X} = \frac{s}{\sqrt n}\sqrt{1 - \frac{n}{N}}$, $s_{\hat p} = \sqrt{\frac{\hat p \hat q}{n-1}}\sqrt{1 - \frac{n}{N}}$.

Standard errors for IID sampling $s_{\bar X} = \frac{s}{\sqrt n}$, $s_{\hat p} = \sqrt{\frac{\hat p \hat q}{n-1}}$

3 Confidence intervals

Approximate sampling distribution $\bar X \overset{a}{\sim} \mathrm{N}(\mu, \frac{\sigma^2}{n})$

$\mathrm{P}(\bar X - z s_{\bar X} < \mu < \bar X + z s_{\bar X}) = \mathrm{P}(-z < \frac{\bar X - \mu}{s_{\bar X}} < z) \approx 2(1 - \Phi(z))$

Approximate 100(1–α)% two-sided CI for μ and p: $\bar X \pm z_{\alpha/2} \cdot s_{\bar X}$ and $\hat p \pm z_{\alpha/2} \cdot s_{\hat p}$, if n is large

100(1–α)%	68%	80%	90%	95%	99%	99.7%
$z_{\alpha/2}$	1.00	1.28	1.64	1.96	2.58	3.00

The higher is confidence level the wider is the CI, the larger is sample the narrower is the CI.

95% CI is a random interval: out of 100 intervals computed for 100 samples
Bin(100,0.95) $\approx$ N(95,$(2.18)^2$) will cover the true value.

4 Stratified random sampling

Population consists of L strata with known L strata fractions $W_1 + \ldots + W_L = 1$ and unknown strata means μ_l and standard deviations σ_l

Population mean $\mu = W_1\mu_1 + \ldots + W_L\mu_L$,
population variance $\sigma^2 = \overline{\sigma^2} + \sum W_l(\mu_l - \mu)^2$,
average variance $\overline{\sigma^2} = W_1\sigma_1^2 + \ldots + W_L\sigma_L^2$,
average standard deviation $\bar{\sigma} = W_1\sigma_1 + \ldots + W_L\sigma_L$.

Stratified random sampling: take L independent samples from each stratum with sample means $\bar{X}_1, \ldots, \bar{X}_L$

$$\boxed{\text{Stratified sample mean: } \bar{X}_s = W_1\bar{X}_1 + \ldots + W_L\bar{X}_L}$$

$\bar{X}_s$ is an unbiased and consistent estimate of μ: $\mathrm{E}(\bar{X}_s) = W_1\mathrm{E}(\bar{X}_1) + \ldots + W_L\mathrm{E}(\bar{X}_L) = \mu$.
Sample variance $s^2_{\bar{X}_s} = (W_1 s_{\bar{X}_1})^2 + \ldots + (W_L s_{\bar{X}_L})^2$

$$\boxed{\text{Approximate CI for } \mu\text{: } \bar{X}_s \pm z_{\alpha/2} \cdot s_{\bar{X}_s}}$$

Pooled sample mean $\bar{X}_p = \frac{1}{n}(n_1\bar{X}_1 + \ldots + n_L\bar{X}_L)$, polled sample size $n = n_1 + \ldots + n_L$.
$\mathrm{E}(\bar{X}_p) = \frac{n_1}{n}\mu_1 + \ldots + \frac{n_L}{n}\mu_L = \mu + \sum(\frac{n_l}{n} - W_l)\mu_l$,
$\mathrm{bias}(\bar{X}_p) = \sum(\frac{n_l}{n} - W_l)\mu_l$.

Example. Students heights: $L = 2$, $W_1 = W_2 = 0.5$, compare $\bar{X}_s$ with $\bar{X}_p$.

$$\boxed{\text{Optimal allocation: } n_l = n\frac{W_l\sigma_l}{\bar{\sigma}},\ \mathrm{Var}(\bar{X}_{so}) = \frac{1}{n} \cdot \bar{\sigma}^2}$$

$\bar{X}_{so}$ minimizes standard error of X_s. Weakness: usually unknown σ_l and $\bar{\sigma}$.

$$\boxed{\text{Proportional allocation: } n_l = nW_l,\ \mathrm{Var}(\bar{X}_{sp}) = \frac{1}{n} \cdot \overline{\sigma^2}}$$

Compare three unbiased estimates of μ: $\mathrm{Var}(\bar{X}_{so}) \leq \mathrm{Var}(\bar{X}_{sp}) \leq \mathrm{Var}(\bar{X})$.
Variability in σ_l across strata:

$$\mathrm{Var}(\bar{X}_{sp}) - \mathrm{Var}(\bar{X}_{so}) = \frac{1}{n}(\overline{\sigma^2} - \bar{\sigma}^2) = \frac{1}{n}\sum W_l(\sigma_l - \bar{\sigma})^2.$$

Variability in means μ_l across strata:

$$\mathrm{Var}(\bar{X}) - \mathrm{Var}(\bar{X}_{sp}) = \frac{1}{n}(\sigma^2 - \overline{\sigma^2}) = \frac{1}{n}\sum W_l(\mu_l - \mu)^2.$$