Serik Sagitov, Chalmers and GU, February 14, 2018

Solutions chapter 11

Problem 11.1

Four random numbers generated from a normal distribution

$$X_1 = 1.1650, \quad X_2 = 0.6268, \quad X_3 = 0.0751, \quad X_4 = 0.3516,$$

along with five random numbers with the same variance σ^2 but perhaps a different mean

$$Y_1 = 0.3035, \quad Y_2 = 2.6961, \quad Y_3 = 1.0591, \quad Y_4 = 2.7971, \quad Y_4 = 1.2641.$$

(a) $\bar{X} = 0.5546$, $\bar{Y} = 1.6240$, $\bar{Y} - \bar{X} = 1.0694$

(b) We have $s_x^2 = 0.2163$, $s_y^2 = 1.1795$, $s_p^2 = 0.7667$. The latter is an unbiased estimate of σ^2.

(c) $s_{\bar{y}-\bar{x}} = 0.5874$

(d) Based on t_7-distribution, an exact 90% CI for $(\mu_y - \mu_x)$ is 1.0694 ± 1.1128.

(e) More appropriate to use a two-sided test.

(f) From the observed test statistic value $T = 1.8206$, we find the two-sided $P = 0.1115$ using the Matlab command2*tcdf(-1.8206,7).

(g) No, because the P-value is larger than 0.01.

(h) Given $\sigma^2 = 1$, we answer differently to some of the the above questions:

b: $\sigma^2 = 1$,
c: $s_{\bar{y}-\bar{x}} = 0.0.6708$,
d: 1.0694 ± 1.1035,
f: $Z = 1.5942$ two-sided $P = 0.11$.

Problem 11.3

In the "two independent samples" setting we have two ways of estimating the variance of $\bar{X} - \bar{Y}$:

(a) $s_p^2(\frac{1}{n} + \frac{1}{m})$, if $\sigma_x = \sigma_y$,
(b) $\frac{s_x^2}{n} + \frac{s_y^2}{m}$ without the assumption of equal variances.

If $m = n$, then these two estimates are identical:

$$s_p^2\Big(\frac{1}{n} + \frac{1}{m}\Big) = \frac{2}{n} \cdot \frac{\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^n (Y_i - \bar{Y})^2}{2n-2} = \frac{s_x^2 + s_y^2}{n} = \frac{s_x^2}{n} + \frac{s_y^2}{m}.$$

Problem 11.8

Test the null hypothesis of no drug effect

H_0 : the drug is not effective for reducing high blood pressure,

using the Mann-Whitney = Wilcoxon rank sum test.

Suggested measurement design: during the same 10 days take measurements on 4 people, two on the treatment X, X', and two controls Y, Y':

$$\begin{aligned} &X_1, \ldots, X_{10};\\ &X'_1, \ldots, X'_{10};\\ &Y_1, \ldots, Y_{10};\\ &Y'_1, \ldots, Y'_{10}. \end{aligned}$$

Dependencies across the days and the people. Proper design of two independent samples: 20 people on the treatment and 20 controls:

$$\begin{aligned} &X_1, \ldots, X_{20};\\ &Y_1, \ldots, Y_{20}. \end{aligned}$$

Problem 11.13

Let $X_1, \ldots, X_{25}$ be IID from $\mathrm{N}(0.3, 1)$. Consider testing at $\alpha = 0.05$

$$H_0 : \mu = 0, \qquad H_1 : \mu > 0.$$

(a) Normal distribution model $X \sim \mathrm{N}(\mu, 1)$. Since $\bar{X} \sim \mathrm{N}(\mu, 1/25)$, we reject H_0 for

$$5\bar{X} > 1.645, \qquad \bar{X} > 0.33.$$

We know the true value $\mu = 0.3$. The power of the test

$$1 - \beta = \mathrm{P}_{H_1}(\bar{X} > 0.33) = 1 - \Phi\Big(\frac{0.33 - 0.3}{1/5}\Big) = 1 - \Phi(0.15) = 0.44.$$

(b) The sign test statistic

$$T = \text{number of positive } X_i, \qquad T \overset{H_0}{\sim} \mathrm{Bin}(25, \tfrac{1}{2}) \approx \mathrm{N}(\tfrac{25}{2}, \tfrac{25}{4}).$$

Reject H_0 for $T \geq k$, where

$$0.05 = \mathrm{P}_{H_0}(T \geq k) = \mathrm{P}_{H_0}(T > k - 1) \approx 1 - \Phi\Big(\frac{k - 0.5 - 12.5}{5/2}\Big) = 1 - \Phi\Big(\frac{k - 13}{2.5}\Big),$$

so that

$$\frac{k - 13}{2.5} = 1.645, \qquad k = 17.$$

With $\mu = 0.3$, we have

$$\mathrm{P}_{H_1}(X > 0) = 1 - \Phi(-0.3) = \Phi(0.3) = 0.62,$$

and

$$T \overset{H_1}{\sim} \mathrm{Bin}(25, 0.62) \approx \mathrm{N}(15.5, 5.89).$$

Te power of the sign test

$$1 - \beta = \mathrm{P}_{H_1}(T \geq 17) = 1 - \Phi\Big(\frac{17 - 0.5 - 15.5}{2.4}\Big) = 1 - \Phi(0.41) = 0.34$$

is lower.

Problem 11.15

Two independent samples of of equal size n are taken from two population distributions with equal standard deviation $\sigma = 10$. Approximate 95% CI for $(\mu_x - \mu_y)$ is

$$\bar{X} - \bar{Y} \pm 1.96 \cdot 10 \cdot \sqrt{\frac{2}{n}}.$$

If the CI has width $2 = 55.44/\sqrt{n}$, then $n \approx 768$.

Problem 11.21

Data: millions of cycles until failure for two types of engine bearings

	Rank	Type I	Type II	Rank
	1	3.03	3.19	2
	8	5.53	4.26	3
	9	5.60	4.47	4
	11	9.30	4.53	5
	13	9.92	4.67	6
	14	12.51	4.69	7
	17	12.95	6.79	10
	18	15.21	9.37	12
	19	16.04	12.75	15
	20	16.84	12.78	16
Rank sum	130			80

Test the null hypothesis of no difference against the two-sided alternative

$$H_0 : \mu_x = \mu_y, \qquad H_1 : \mu_x \neq \mu_y.$$

(a) Two-sample t-test

$$\bar{X} = 10.693, \quad \bar{Y} = 6.750, \quad s_x^2 = 23.226, \quad s_y^2 = 12.978, \quad s_{\bar{x}-\bar{y}} = \sqrt{s_{\bar{x}}^2 + s_{\bar{y}}^2} = 1.903.$$

Assume equal variances. The observed test statistic

$$T = \frac{10.693 - 6.750}{1.903} = 2.072.$$

With df = 18, the two-sided $P = 0.053$ is found using the Matlab command 2*tcdf(-2.072,18).

(b) Wilcoxon rank sum test statistics $R_x = 130$, $R_y = 80$. From the table on page A22 we find that the two-sided P-value is between $0.05 < P < 0.10$.

(c) The non-parametric test in (b) is more relevant, since both normplot(x) and normplot(y) show non-normality of the data distribution.

(d) To estimate the probability π, that a type I bearing will outlast a type II bearing, we turn to the ordered pooled sample

X-YYYYYY-XX-Y-X-Y-XX-YY-XXXX.

Pick a pair (X, Y) at random, then by the division rule of probability

$$\mathrm{P}(X < Y) = \frac{\text{number of } (x_i < y_j)}{\text{total number of pairs } (x_i, y_j)} = \frac{10 + 4 + 4 + 3 + 2 + 2}{100} = 0.25.$$

This implies a point estimate $\hat{\pi} = 0.75$.

(e) The matlab commands

```
u=x(random('unid',10,10,1000));
v=y(random('unid',10,10,1000));
N=zeros(1,1000);
for k=1:1000 for i=1:10 for j=1:10
N(k)=N(k)+(u(i,k)>v(j,k));
end,end,end
P=N/100;
hist(P,20)
std(P)
```

estimate the sampling distribution of $\hat{\pi}$ with $s_{\hat{\pi}} = 0.1187$.

(f) The Matlab commands

```
c1=prctile(P,5)
c2=prctile(P,95)
```

give a 90% CI for π: $(2\hat{\pi} - c2; 2\hat{\pi} - c1) = (0.58; 0.96)$.

Problem 11.28

Two-sided signed rank test. For $n = 10, 20, 25$ and $\alpha = 0.05, 0.01$, compare the critical values from the table and using the normal approximation of the null distribution. Using

$$W_{0.05}(n) = \frac{n(n+1)}{4} - 1.96 \cdot \sqrt{\frac{n(n+1)(2n+1)}{24}},$$
$$W_{0.01}(n) = \frac{n(n+1)}{4} - 2.58 \cdot \sqrt{\frac{n(n+1)(2n+1)}{24}},$$

we find (table/normal approximation)

	$n=10$	$n=20$	$n=25$
$\frac{n(n+1)}{4}$	27.5	105	162.5
$\sqrt{\frac{n(n+1)(2n+1)}{24}}$	9.81	26.79	37.17
$\alpha = 0.05$	8/8.3	52/53.5	89/89.65
$\alpha = 0.01$	3/2.2	38/36.0	68/67.6

Problem 11.27

Find the exact null distribution for the test statistic of the signed rank test with $n = 4$.

Model: IID differences $D_1, \ldots, D_n$ whose population distribution is symmetric around the unknown median M. Test the null hypothesis of no difference $H_0 : M = 0$ using the signed ranks defined as follows:

step 1: remove signs $|D_1|, \ldots, |D_n|$,
step 2: assign ranks $1, \ldots, n$ to $|D_1|, \ldots, |D_n|$,
step 3: attach accordingly the original signs to the ranks $1, \ldots, n$,
step 4: compute W_+ as the sum of the positive ranks.

Under $H_0 : M = 0$, on the step 4, the signs $\pm$ are assigned symmetrically at random. There are 16 equally likely outcomes

1	2	3	4	W_+
−	−	−	−	0
+	−	−	−	1
−	+	−	−	2
+	+	−	−	3
−	−	+	−	3
+	−	+	−	4
−	+	+	−	5
+	+	+	−	6
−	−	−	+	4
+	−	−	+	5
−	+	−	+	6
+	+	−	+	7
−	−	+	+	7
+	−	+	+	8
−	+	+	+	9
+	+	+	+	10

Thus the null distribution is given by the table

k	0	1	2	3	4	5	6	7	8	9	10
p_k	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$

The smallest one-sided P-value is $\frac{1}{16} = 0.06$.

Problem 11.34

Two population distributions with $\sigma_x = \sigma_y = 10$. Two samples of sizes $n = 25$ can be taken in two ways

(a) paired with $\text{Cov}(X_i, Y_i) = 50$, $i = 1, \ldots, 25$,
(b) unpaired $X_1, \ldots, X_{25}$ and $Y_1, \ldots, Y_{25}$.

Compare the power curves for testing

$$H_0 : \mu_x = \mu_y, \qquad H_1 : \mu_x > \mu_y, \qquad \alpha = 0.05.$$

(a) The variance of a difference

$$\text{Var}(D) = \text{Var}(X - Y) = \sigma_x^2 + \sigma_y^2 - 2\text{Cov}(X, Y) = 100 + 100 - 100 = 100.$$

Using the normal approximation we get

$$\bar{D} = \bar{X} - \bar{Y} \approx \text{N}(\mu_x - \mu_y, \tfrac{100}{25}) = \text{N}(\delta, 4).$$

The rejection region $\{\bar{D} > 2 \cdot 1.645 = 3.29\}$. The power function

$$\text{Pw}(\delta) = \text{P}(\bar{D} > 3.29) \approx 1 - \Phi(\tfrac{3.29-\delta}{2}).$$

(b) Two independent samples

$$\bar{X} - \bar{Y} \approx \text{N}(\mu_x - \mu_y, \tfrac{100}{25} + \tfrac{100}{25}) = \text{N}(\delta, 8).$$

The rejection region $\{\bar{X} - \bar{Y} > \sqrt{8} \cdot 1.645 = 4.65\}$. The power function

$$\text{Pw}(\delta) = \text{P}(\bar{X} - \bar{Y} > 4.65) \approx 1 - \Phi(\tfrac{4.65-\delta}{2.83}).$$

See Figure 1.

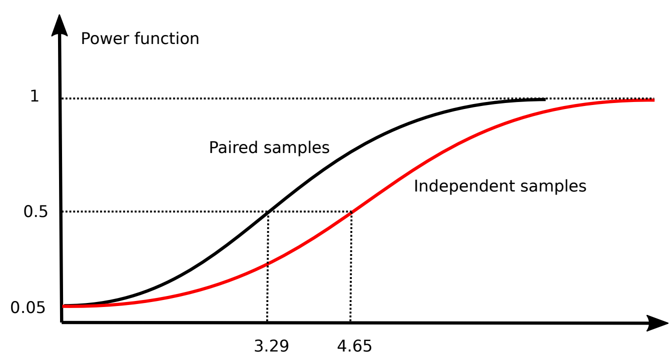

Figure 1: Two power functions of Problem 11.34. More power with paired sample design.

Problem 11.36

Paired samples

$$\begin{aligned}
&\bar{X} = 85.26, \quad s_x = 21.20, \quad s_{\bar{x}} = 5.47, \quad n = 15, \\
&\bar{Y} = 84.82, \quad s_y = 21.55, \quad s_{\bar{y}} = 5.57, \quad m = 15, \\
&\bar{D} = \bar{X} - \bar{Y} = 0.44, \\
&s_d = 4.63, \quad s_{\bar{x}-\bar{y}} = 1.20.
\end{aligned}$$

If the pairing had been erroneously ignored, then the two independent samples formula would give 6 times larger standard error

$$s_{\bar{x}-\bar{y}} = 7.81.$$

To test $H_0 : \mu_x = \mu_y$ against $H_1 : \mu_x \neq \mu_y$ assume $D \sim \mathrm{N}(\mu, \sigma^2)$ and apply one-sample t-test

$$T = \frac{\bar{D}}{s_{\bar{d}}} = 0.368.$$

With df = 14, two-sided $P = 0.718$, we can not reject H_0.
Without normality assumption apply the Wilcoxon signed rank test. Matlab command

```
signrank(x,y)
```

computes the two-sided $P = 0.604$. We can not reject H_0.

Problem 11.52

Possible explanations

(a) room with a window $\leftarrow$ rich patient $\rightarrow$ recovers faster,
(b) smoker $\leftarrow$ the man is a loser $\rightarrow$ wife gets cancer,
(c) no breakfast $\leftarrow$ lack of discipline $\rightarrow$ accident,
(d) choose to change the school $\leftarrow$ lower grades before $\rightarrow$ lower grades after,
(e) match two babies with two mothers,
(f) abstain from alcohol $\leftarrow$ poor health,
(g) marijuana $\leftarrow$ schizophrenia,
(h) total time together = time before wedding + time after wedding,
(i) better physical condition $\rightarrow$ attend church.