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Solutions chapter 12

Matlab commands:

```
x = data matrix
boxplot(x)
anova1(x)
anova2(x)
```

Problem 12.3

Consider one-way ANOVA test statistic

$$F = \frac{MS_A}{MS_E} = \frac{\frac{J}{I-1}\sum_{i=1}^{I}(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2}{\frac{1}{I(J-1)}\sum_{i=1}^{I}\sum_{j=1}^{J}(Y_{ij} - \bar{Y}_{i\cdot})^2}$$

For $I = 2$ and $J = n$, put

$$\bar{Y}_{1\cdot} = \bar{X}, \quad \bar{Y}_{2\cdot} = \bar{Y}, \quad \bar{Y}_{\cdot\cdot} = \frac{\bar{X} + \bar{Y}}{2}.$$

In this two-sample setting, the F-test statistic

$$F = \frac{n[(\bar{X} - \frac{\bar{X}+\bar{Y}}{2})^2 + (\bar{Y} - \frac{\bar{X}+\bar{Y}}{2})^2]}{\frac{1}{2(n-1)}[\sum_{j=1}^{n}(X_j - \bar{X})^2 + \sum_{j=1}^{n}(Y_j - \bar{Y})^2]} = \frac{2n(\frac{\bar{X}-\bar{Y}}{2})^2}{s_p^2} = (\tfrac{\bar{X}-\bar{Y}}{s_p\sqrt{\frac{2}{n}}})^2,$$

equals T^2, where $T = \frac{\bar{X}-\bar{Y}}{s_p\sqrt{\frac{2}{n}}}$ is the two-sample t-test statistic.

Problem 12.5

Derive the likelihood ratio test for the one-way layout and show that it is equivalent to the F-test.

The null hypothesis says that the data Y_{ij} comes from a single normal distribution

$$H_0 : \mu_1 = \ldots = \mu_I = \mu$$

described by two parameters μ and σ^2, so that $\dim \Omega_0 = 2$, while $\dim \Omega = I + 1$. The likelihood ratio

$$\Lambda = \frac{L_0(\hat{\mu}, \hat{\sigma}_0^2)}{L(\hat{\mu}_1, \ldots, \hat{\mu}_I, \hat{\sigma}^2)},$$

where putting $n = IJ$,

$$L(\mu_1, \ldots, \mu_I, \sigma^2) = \prod_{i=1}^{I}\prod_{j=1}^{J} \tfrac{1}{\sqrt{2\pi}\sigma} e^{\frac{(Y_{ij}-\mu_i)^2}{2\sigma^2}} \propto \sigma^{-n}\exp\{-\Sigma\Sigma\tfrac{(Y_{ij}-\mu_i)^2}{2\sigma^2}\},$$

$$L_0(\mu, \sigma^2) = L(\mu, \ldots, \mu, \sigma^2) \propto \sigma^{-n}\exp\{-\Sigma\Sigma\tfrac{(Y_{ij}-\mu)^2}{2\sigma^2}\}.$$

We find the maximum likelihood estimates to be

$$\hat{\mu} = \bar{Y}_{..}, \quad \hat{\sigma}_0^2 = \tfrac{SS_T}{n}, \quad \hat{\mu}_i = \bar{Y}_{i.}, \quad \hat{\sigma}^2 = \tfrac{SS_E}{n},$$

which yields

$$\Lambda = \frac{\hat{\sigma}_0^{-n} \exp\{-\Sigma\Sigma \frac{(Y_{ij}-\hat{\mu})^2}{2\hat{\sigma}_0^2}\}}{\hat{\sigma}^{-n} \exp\{-\Sigma\Sigma \frac{(Y_{ij}-\hat{\mu}_i)^2}{2\hat{\sigma}^2}\}} = (\tfrac{\hat{\sigma}_0^2}{\hat{\sigma}^2})^{-n/2}.$$

The likelihood ratio test rejects the null hypothesis for small values of Λ or equivalently for large values of

$$\tfrac{\hat{\sigma}_0^2}{\hat{\sigma}^2} = \tfrac{SS_T}{SS_E} = 1 + \tfrac{SS_A}{SS_E} = 1 + \tfrac{J(I-1)MS_A}{I(J-1)MS_E} = 1 + \tfrac{J(I-1)}{I(J-1)} \cdot F$$

that is for large values of F-test statistics. This leads to an asymptotic approximation of the $F_{J(I-1),I(J-1)}$ in terms of the chi-square distribution with df $= I - 1$.

Problem 12.10

One-way layout with $I = 10$, $J = 7$, $X_{ij} \sim \mathrm{N}(\mu_i, \sigma^2)$. Pooled sample variance

$$s_p^2 = \mathrm{MS_E} = \frac{1}{I(J-1)} \sum_i \sum_j (X_{ij} - \bar{X}_{i.})^2$$

uses df $= I(J-1) = 60$.

(a) A 95% CI for a single difference $\mu_u - \mu_v$

$$\bar{X}_{u\cdot} - \bar{X}_{v\cdot} \pm t_{60}(0.025) s_p \sqrt{\tfrac{2}{J}}$$

has the half-width of

$$2.82 \cdot \tfrac{s_p}{\sqrt{J}}.$$

(b) Bonferroni simultaneous 95% CI for $\binom{10}{2} = 45$ differences $\mu_u - \mu_v$

$$\bar{X}_{u\cdot} - \bar{X}_{v\cdot} \pm t_{60}(\tfrac{0.025}{45}) s_p \sqrt{\tfrac{2}{J}}$$

has the half-width of

$$4.79 \cdot \tfrac{s_p}{\sqrt{J}},$$

giving the ratio

$$\tfrac{4.79}{2.82} = 1.7.$$

(c) Tukey simultaneous 95% CI for differences $\mu_u - \mu_v$

$$\bar{X}_{u\cdot} - \bar{X}_{v\cdot} \pm q_{10,60}(0.05) \tfrac{s_p}{\sqrt{J}}$$

has the half-width of

$$4.65 \cdot \tfrac{s_p}{\sqrt{J}},$$

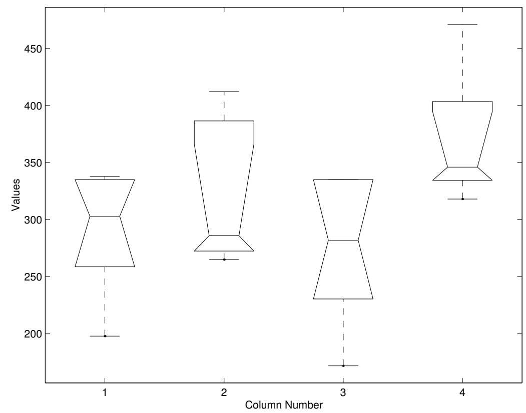
giving the ratio

$$\frac{\text{Bonferroni}}{\text{Tukey}} = \tfrac{4.79}{4.65} = 1.03.$$

Problem 12.21

For $I = 4$ control groups of $J = 5$ mice each, test H_0: no systematic differences between groups.

> Significant differences among the control groups, although not expected, might be attributable to changes in the experimental conditions.



One way ANOVA table

Source	SS	df	MS	F	P
Columns	27230	3	9078	2.271	0.12
Error	63950	16	3997		
Total	91190	19			

Do not reject H_0 at 10% significance level. Boxplots show non-normality. The largest difference is between the third and the fourth boxplots. Control question: why the third boxplot has no upper whisker?

Kruskal-Wallis test. Pooled sample ranks

group I	2	6	9	11	14	$\bar{R}_{1.} = 8.4$
group II	4	5	8	17	19	$\bar{R}_{2.} = 10.6$
group III	1	3	7	12.5	12.5	$\bar{R}_{3.} = 7.2$
group IV	10	15	16	18	20	$\bar{R}_{4.} = 15.8$

Kruskal-Wallis test statistic

$$K = \frac{12 \cdot 5}{20 \cdot 21}\Big((8.4 - 10.5)^2 + (10.6 - 10.5)^2 + (7.2 - 10.5)^2 + (15.8 - 10.5)^2\Big) = 6.20.$$

Since $\chi^2_3(0.1) = 6.25$, we do not reject H_0 at 10% significance level.

Problem 12.26

$I = 3$ treatments on $J = 10$ subjects with $K = 1$ observations per cell.

H_0: no treatment effects.

Results of anova2(x):

Source	SS	df	MS	F
Columns (blocks)	0.517	9	0.0574	0.4683
Rows (treatments)	1.081	2	0.5404	4.406
Error	2.208	18	0.1227	
Total	3.806	29		

Two P-values: columns = 0.8772, rows = 0.0277. Reject H_0 at 5% significance level.

Friedman's test. Ranking within blocks:

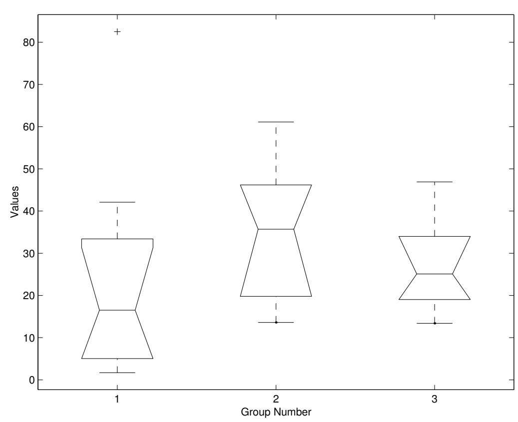
1	2	3	2	1	2	1	3	1	3	$\bar{R}_{1.} = 1.9$
2	1	1	3	2	1	3	1	2	2	$\bar{R}_{2.} = 1.8$
3	3	2	1	3	3	2	2	3	1	$\bar{R}_{3.} = 2.3$

The observed value of the Friedman test statistic

$$Q = \frac{12 \cdot 10}{3 \cdot 4} \left((1.8 - 2)^2 + (1.9 - 2)^2 + (2.3 - 2)^2 \right) = 1.4.$$

Since $\chi_2^2(0.1) = 4.61$, we can not reject H_0 even at 10% significance level.

Problem 12.28



$I = 3$ types of stopwatches, different sample sizes.
H_0: no systematic differences between groups.

One way ANOVA table

Source	SS	df	MS	F
Columns	446.6	2	223.3	0.4974
Error	7632	17	449	
Total	8079	19		

gives the P-value of 0.6167. We do not reject H_0.

Kruskal-Wallis test. Pooled sample ranks

group I: 1, 2, 3, 4, 7, 10.5, 14, 15, 20,	$\bar{R}_{1.} = 8.5$
group II: 6, 8, 12, 16.5, 16.5, 19,	$\bar{R}_{2.} = 13.0$
group III: 5, 9, 10.5, 13, 18,	$\bar{R}_{3.} = 11.1$

The observed value of the test statistic

$$K = \frac{12}{20 \cdot 21} \left(9 \cdot (8.5 - 10.5)^2 + 6 \cdot (13.0 - 10.5)^2 + 5 \cdot (11.1 - 10.5)^2\right) = 2.15.$$

Since $\chi_2^2(0.1) = 4.61$, we do not reject H_0 even at 10% significance level.

Problem 12.34

Forty eight survival times: $I = 3$ poisons and $J = 4$ treatments with $K = 4$ observations per cell. Cell means for the survival times

	A	B	C	D
I	4.125	8.800	5.675	6.100
II	3.200	8.150	3.750	6.625
III	2.100	3.350	2.350	3.250

Draw three profiles: I and II cross each other, and profile III is more flat. Three null hypotheses of interest

H_A: no poison effect,
H_B: no treatment effect,
H_{AB}: no interaction.

(a) Survival in hours x data matrix. Results of anova2(x,4)

Source	SS	df	MS	F
Columns (treatments)	91.9	3	30.63	14.01
Rows (poisons)	103	2	51.52	23.57
Intercation	24.75	6	4.124	1.887
Error	78.69	36	2.186	
Total	298.4	47		

Three P-values: columns = 0.0000, rows = 0.0000, interaction = 0.1100. Reject H_A and H_B at 1% significance level, we can not reject H_{AB} even at 10% significance level:

3 poisons act differently,
4 treatments act differently,
some indication of interaction.

Analysis of the residuals $Y_{ijk} - \bar{Y}_{ij.}$

normal probability plot reveals non-normality,
skewness = 0.59,
kurtosis = 4.1.

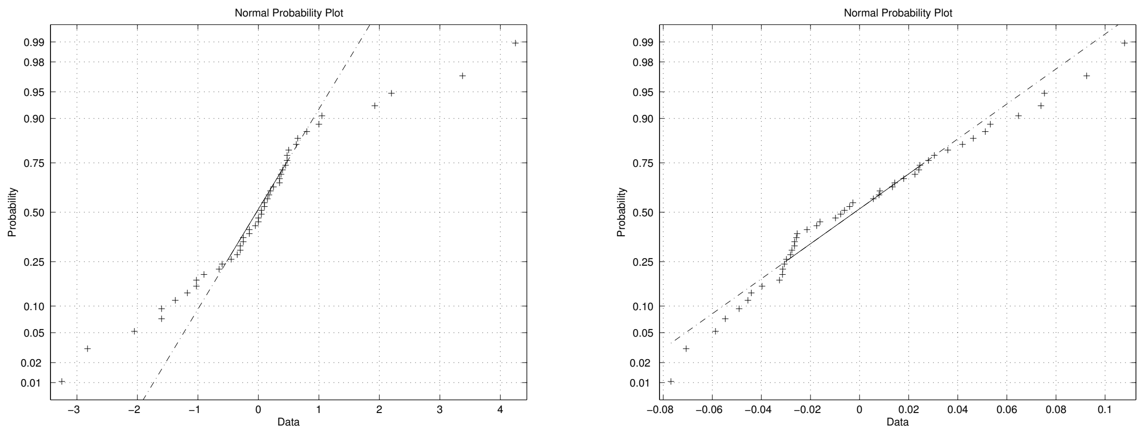


Figure 1: Left panel: survival times. Right panel: death rates.

(b) Transformed data: death rate = 1/survival time. Cell means for the death rates

	A	B	C	D
I	0.249	0.116	0.186	0.169
II	0.327	0.139	0.271	0.171
III	0.480	0.303	0.427	0.309

Draw three profiles: they look more parallel.

New data matrix y=x.ˆ(-1). Results of anova2(y,4):

Source	SS	df	MS	F
Columns (treatments)	0.204	3	0.068	28.41
Rows (poisons)	0.349	2	0.174	72.84
Intercation	0.01157	6	0.0026	1.091
Error	0.086	36	0.0024	
Total	0.6544	47		

Three P-values: columns = 0.0000, rows = 0.0000, interaction = 0.3864. Reject H_A and H_B at 1% significance level, accept H_{AB} at 10% significance level. Conclusions

3 poisons act differently,
4 treatments act differently,
no interaction,
the normal probability plot of residuals reveals a closer fit to normality assumption.