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Solutions chapter 9

Problem 9.3

Data $X \sim \text{Bin}(100, p)$. A pair of alternatives

$$H_0 : p = 1/2, \quad H_1 : p \neq 1/2.$$

Consider the test that rejects H_0 for $|X - 50| > 10$. The z-score

$$Z = \frac{X - 100p}{10\sqrt{p(1-p)}}$$

distribution is approximately $\text{N}(0, 1)$.

(a) The corresponding significance level is

$$\alpha = \text{P}_{H_0}(|X - 50| > 10) = \text{P}_{H_0}(|Z| > 2) \approx 2(1 - \Phi(2)) = 2 \cdot 0.0228 = 0.046.$$

(b) The power of the test is a function of the parameter value p

$$\begin{aligned}\text{Pw}(p) &= \text{P}(|X - 50| > 10) = \text{P}(X < 40) + \text{P}(X > 60)\\ &= \text{P}\left(Z < \frac{40 - 100p}{10\sqrt{p(1-p)}}\right) + \text{P}\left(Z > \frac{60 - 100p}{10\sqrt{p(1-p)}}\right)\\ &= \Phi\left(\frac{4 - 10p}{\sqrt{p(1-p)}}\right) + \Phi\left(\frac{10p - 6}{\sqrt{p(1-p)}}\right).\end{aligned}$$

Putting $\delta = 1/2 - p$, we see that the power function

$$\text{Pw}(p) = \Phi\left(\frac{10\delta - 1}{\sqrt{1/4 - \delta^2}}\right) + \Phi\left(-\frac{10\delta + 1}{\sqrt{1/4 - \delta^2}}\right)$$

is symmetric around $p = 1/2$

p	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70
$\text{Pw}(p)$	0.986	0.853	0.500	0.159	0.046	0.159	0.500	0.853	0.986

Problem 9.4

Two distributions for X

X-values	x_1	x_2	x_3	x_4
$\text{P}(x\|H_0)$	0.2	0.3	0.3	0.2
$\text{P}(x\|H_1)$	0.1	0.4	0.1	0.4
Likelihood ratio $\Lambda = \frac{\text{P}(x\|H_0)}{\text{P}(x\|H_1)}$	2	0.75	3	0.5

Data: one observation of X. Likelihood ratio test: reject for small values of Λ.

(a) See the bottom line of the table.

(b) The null distribution of Λ

X-values	x_4	x_2	x_1	x_3
Likelihood ratio Λ	0.5	0.75	2	3
$\mathrm{P}(x\|H_0)$	0.2	0.3	0.2	0.3
Cumulative probab.	0.2	0.5	0.7	1

At $\alpha = 0.2$ we reject H_0 only if $\Lambda = 0.5$ that is when $X = x_4$.
At $\alpha = 0.5$ we reject H_0 for $\Lambda \leq 0.75$ that is when $X = x_4$ or x_2.

(c) If the prior probabilities are $\mathrm{P}(H_0) = \mathrm{P}(H_1) = \frac{1}{2}$, which outcomes favour H_0? By Bayes formula,

$$\mathrm{P}(H_0|x) = \frac{\mathrm{P}(x|H_0)\mathrm{P}(H_0)}{\mathrm{P}(x|H_0)\mathrm{P}(H_0) + \mathrm{P}(x|H_1)\mathrm{P}(H_1)} = \frac{\mathrm{P}(x|H_0)}{\mathrm{P}(x|H_0) + \mathrm{P}(x|H_1)}, \quad x = x_1, x_2, x_3, x_4.$$

Thus the posterior odds ratio equals the likelihood ratio

$$\frac{\mathrm{P}(H_0|x)}{\mathrm{P}(H_1|x)} = \Lambda,$$

and we conclude that outcomes x_1 and x_3 favour H_0 (assuming equal costs with these outcomes we have $\Lambda > 1$).

(d) For the general prior

$$\mathrm{P}(H_0) = \pi_0, \quad \mathrm{P}(H_1) = \pi_1 = 1 - \pi_0,$$

we get

$$\mathrm{P}(H_i|x) = \frac{\mathrm{P}(x|H_i)\pi_i}{\mathrm{P}(x|H_0)\pi_0 + \mathrm{P}(x|H_1)\pi_1}, \quad \frac{\mathrm{P}(H_0|x)}{\mathrm{P}(H_1|x)} = \frac{\mathrm{P}(x|H_0)\pi_0}{\mathrm{P}(x|H_1)\pi_1} = \Lambda \cdot \frac{\pi_0}{\pi_1}.$$

Assuming equal costs, the rejection rule is $\frac{\mathrm{P}(H_0|x)}{\mathrm{P}(H_1|x)} < 1$, so that in terms of the likelihood ratio,

$$\Lambda < \frac{\pi_1}{\pi_0} = \frac{1}{\pi_0} - 1, \qquad \pi_0 < \frac{1}{1+\Lambda}.$$

If $x = x_4$, then $\Lambda = 0.5$, and we reject H_0, provided $\pi_0 < \frac{2}{3}$.
If $x = x_2$, then $\Lambda = 0.75$, and we reject H_0, provided $\pi_0 < \frac{4}{7}$.
If $x = x_1$, then $\Lambda = 2$, and we reject H_0, provided $\pi_0 < \frac{1}{3}$.
If $x = x_3$, then $\Lambda = 3$, and we reject H_0, provided $\pi_0 < \frac{1}{4}$.

Problem 9.7

Likelihood function

$$L(\lambda) = e^{-\lambda n} \lambda^{x_1+\ldots+x_n} \prod_{i=1}^{n} \frac{1}{x_i!}.$$

Reject H_0 for small

$$\frac{L(\lambda_0)}{L(\lambda_1)} = e^{-n(\lambda_0-\lambda_1)} (\frac{\lambda_0}{\lambda_1})^{x_1+\ldots+x_n}.$$

If $\lambda_1 > \lambda_0$, then we reject H_0 for large

$$Y = X_1 + \ldots + X_n.$$

Test statistic Y has null distribution $\mathrm{Pois}(n\lambda_0)$.

Problem 9.9

IID sample from $\mathrm{N}(\mu, 100)$ of size $n = 25$. Two simple hypotheses

$$H_0 : \mu = 0, \quad H_1 : \mu = 1.5$$

Test statistic and its exact sampling distribution

$$\bar{X} \sim \mathrm{N}(\mu, 4).$$

Its null distribution is $\mathrm{N}(0, 2^2)$.

(a) The rejection region at $\alpha = 0.1$ is $\{\bar{X} > x\}$, where x is the solution of the equation

$$0.1 = \mathrm{P}_{H_0}(\bar{X} > x) = 1 - \Phi(x/2).$$

From the normal distribution table we find $x/2 = 1.28$, so that $x = 2.56$.

(b) The power of the test (a) is

$$\mathrm{P}_{H_1}(\bar{X} > 2.56) = \mathrm{P}_{H_1}\Big(\frac{\bar{X} - 1.5}{2} > 0.53\Big) = 1 - \Phi(0.53) = 1 - 0.7019 = 0.298.$$

(c) For $\alpha = 0.01$ the rejection region is $\{\bar{X} > 4.66\}$, since $1 - \Phi(2.33) = 0.01$. The power of this test is

$$\mathrm{P}_{H_1}(\bar{X} > 4.66) = \mathrm{P}_{H_1}\Big(\frac{\bar{X} - 1.5}{2} > 1.58\Big) = 1 - \Phi(1.58) = 1 - 0.9429 = 0.057.$$

Problem 9.14

For a single observation $X \sim \mathrm{N}(\mu, \sigma^2)$, where σ^2 is known, test $H_0 : \mu = 0$ vs $H_1 : \mu = 1$. Prior probabilities

$$\mathrm{P}(H_0) = \frac{2}{3}, \quad \mathrm{P}(H_1) = \frac{1}{3}.$$

Choose H_0 whenever $\mathrm{P}(H_0|x) > \mathrm{P}(H_1|x)$.

(a) Likelihood ratio

$$\frac{f(x|0)}{f(x|1)} = \frac{e^{-\frac{x^2}{2\sigma^2}}}{e^{-\frac{(x-1)^2}{2\sigma^2}}} = e^{\frac{\frac{1}{2}-x}{\sigma^2}}.$$

Choose H_0 for x such that

$$\frac{\mathrm{P}(H_0|x)}{\mathrm{P}(H_1|x)} = 2e^{\frac{\frac{1}{2}-x}{\sigma^2}} > 1, \qquad x < \frac{1}{2} + \sigma^2 \ln 2.$$

(b) In the long run, the proportion of the time H_0 will be chosen is

$$\mathrm{P}(X < \frac{1}{2} + \sigma^2 \ln 2) = \frac{2}{3} \cdot \Phi\Big(\sigma \ln 2 + \frac{1}{2\sigma}\Big) + \frac{1}{3} \cdot \Phi\Big(\sigma \ln 2 - \frac{1}{2\sigma}\Big).$$

In particular, if $\sigma = 1$, then this is 0.78.

Problem 9.22

An exact 95% CI for σ^2 if $n = 15$ is

$$(0.536 s^2; 2.487 s^2).$$

Reject $H_0 : \sigma = 1$

$$\text{if} \quad s^2 > 1.866 \quad \text{or} \quad s^2 < 0.402.$$

Problem 9.23

An IID sample from $\mathrm{N}(\mu, \sigma^2)$ gives a 99% CI for μ to be $(-2, 3)$. Test

$$H_0 : \mu = -3 \quad \text{against} \quad H_1 : \mu \neq -3$$

at $\alpha = 0.01$.

Using the CI-method of hypotheses testing we reject H_0 in favour of the two-sided alternative, since the value $\mu = -3$ is not covered by the two-sided confidence interval $(-2, 3)$.

Problem 9.24

Binomial data

$$X \sim \mathrm{Bin}(n, p).$$

Test

$$H_0 : p = 0.5 \quad \text{against} \quad H_1 : p \neq 0.5$$

(a) Generalised likelihood ratio

$$\Lambda = \frac{L(p_0)}{L(\hat{p})} = \frac{\binom{n}{x}\big(\frac{1}{2}\big)^n}{\binom{n}{x}\big(\frac{x}{n}\big)^x\big(\frac{n-x}{n}\big)^{n-x}} = \frac{\big(\frac{n}{2}\big)^n}{x^x(n-x)^{n-x}}.$$

(b) The generalised likelihood ratio test rejects H_0 for small values of

$$\ln \Lambda = n \ln(n/2) - x \ln x - (n-x)\ln(n-x),$$

or equivalently, for large values of

$$x \ln x + (n-x)\ln(n-x).$$

or equivalently, for large values of

$$l(y) := (n/2+y)\ln(n/2+y) + (n/2-y)\ln(n/2-y),$$

where

$$y = |x - n/2|.$$

The function $l(y)$ is monotonely increasing over $y \in [0, n/2]$, since

$$l'(y) = \ln \frac{\frac{n}{2}+y}{\frac{n}{2}-y} > 0.$$

We conclude that the test rejects for large values of $|X - \frac{n}{2}|$.

(c) Compute the significance level for the rejection region $|X - \frac{n}{2}| > k$:

$$\alpha = \mathrm{P}_{H_0}\Big(|X - \frac{n}{2}| > k\Big) = 2 \sum_{i < \frac{n}{2}-k} \binom{n}{i} 2^{-n}.$$

(d) In particular, for $n = 10$ and $k = 2$ we get

$$\alpha = 2^{-9} \sum_{i=0}^{2} \binom{10}{i} = \frac{1+10+45}{512} = 0.11.$$

(d) Using the normal approximation for $n = 100$ and $k = 10$, we find

$$\alpha = \mathrm{P}_{H_0}(|X - np_0| > k) \approx 2\Big(1 - \Phi\Big(\frac{k}{\sqrt{np_0(1-p_0)}}\Big)\Big) = 2(1-\Phi(2)) = 0.046.$$

Problem 9.28

Observed test statistic $T = 1.50$. Null distribution is a standard normal.

(a) Two-sided P-value = 0.134.

(b) One-sided P-value = 0.067.

Conclusion: choose H_1 before you see your data.

Problem 9.33

California grave yards: data on two weeks around Passover, a Jewish holiday

H_0 : death cannot be postponed,
H_1 : death can be postponed until after an important date.

(a) Jewish data: $n = 1919$ death dates

$Y = 922$ deaths during the week before Passover,
$n - Y = 997$ deaths during the week after Passover.

Under the binomial model $Y \sim \text{Bin}(n, p)$, we would like to test

$$H_0 : p = 0.5 \quad \text{against} \quad H_1 : p < 0.5.$$

We apply the large sample test for proportion. Observed test statistic

$$Z = \frac{922 - 1919 \cdot 0.5}{\sqrt{1919} \cdot 0.5} = -1.712.$$

One-sided P-value of the test

$$\Phi(-1.712) = 1 - \Phi(1.712) = 1 - 0.9564 = 0.044.$$

Reject H_0 in favor of one-sided H_1 at the significance level 5%.

(b) To control for the seasonal effect the Chinese and Japanese data were studied

$$n = 852, \quad Y = 418, \quad n - Y = 434, \quad Z = -0.548.$$

One-sided P-value is 29%, showing no significant effect.

(c) Overeating might be a contributing factor.

Problem 9.35

Multinomial model

$$(X_1, X_2, X_3) \sim \text{Mn}(190, p_1, p_2, p_3).$$

Composite null hypothesis (Hardy-Weinberg Equilibrium)

$$H_0 : p_1 = (1-\theta)^2, p_2 = 2\theta(1-\theta), p_3 = \theta^2.$$

Likelihood function and MLE

$$L(\theta) = \binom{190}{10, 68, 112} 2^{68} \theta^{292} (1-\theta)^{88}, \quad \hat{\theta} = \frac{88}{380} = 0.768.$$

Pearson's chi-square test:

cell	1	2	3	Total
observed	10	68	112	190
expected	10.23	67.71	112.07	190

Observed $X^2 = 0.0065$, df $= 1$, P-value $= 2(1 - \Phi(\sqrt{0.0065})) = 0.94$.

Problem 9.36

US suicides in 1970. Check for the seasonal variation

Month	O_j	Days	E_j	$O_j - E_j$
Jan	1867	31	1994	-127
Feb	1789	28	1801	-12
Mar	1944	31	1994	-50
Apr	2094	30	1930	164
May	2097	31	1994	103
Jun	1981	30	1930	51
Jul	1887	31	1994	-107
Aug	2024	31	1994	30
Sep	1928	30	1930	-2
Oct	2032	31	1994	38
Nov	1978	30	1930	48
Dec	1859	31	1994	-135

Simple null hypothesis

$$H_0:\ p_1 = p_3 = p_5 = p_7 = p_8 = p_{10} = p_{12} = \frac{31}{365},\ p_2 = \frac{28}{365},\ p_4 = p_6 = p_9 = p_{11} = \frac{30}{365}.$$

The total number suicides $n = 23480$, so that the expected counts are

$$E_j = np_j^{(0)}, \quad j = 1, \dots, 12.$$

The χ^2-test statistic

$$X^2 = \sum_j \frac{(O_j - E_j)^2}{E_j} = 47.4.$$

Since df $= 12 - 1 = 11$, and $\chi^2_{11}(0.005) = 26.8$, we reject H_0 of no seasonal variation. Merry Christmas!

Problem 9.43

Number of heads

$$Y \sim \text{Bin}(n, p), \quad n = 17950.$$

(a) For $H_0 : p = 0.5$ the observed $Z = 3.46$. Reject H_0.

(b) Pearson's chi-square test for the simple null hypothesis

$$\begin{aligned} H_0 :& p_0 = (0.5)^5 = 0.031,\ p_1 = 5 \cdot (0.5)^5 = 0.156,\ p_2 = 10 \cdot (0.5)^5 = 0.313, \\ & p_3 = 10 \cdot (0.5)^5 = 0.313,\ p_4 = 5 \cdot (0.5)^5 = 0.156,\ p_5 = (0.5)^5 = 0.031. \end{aligned}$$

number of heads	0	1	2	3	4	5	Total
observed	100	524	1080	1126	655	105	3590
expected	112.2	560.9	1121.9	1121.9	560.9	112.2	3590

Observed $X^2 = 21.58$, df $= 5$, P-value $= 0.001$.

(c) Composite null hypothesis

$$H_0 : p_i = \binom{5}{i} p^i (1-p)^{5-i}, \quad i = 0, 1, 2, 3, 4, 5.$$

Pearson's chi-square test based on the MLE $\hat{p} = 0.5129$

number of heads	0	1	2	3	4	5	Total
observed	100	524	1080	1126	655	105	3590
expected	98.4	518.3	1091.5	1149.3	605.1	127.4	3590

Observed $X^2 = 8.74$, df $= 4$, P-value $= 0.07$. Do not reject H_0 at 5% level.