This solution to compotational task has been taken from anothe course (TMS165/MSA350). In that course Wiener process is denoted $B(t)$ instead of $W(t)$ as in our course.

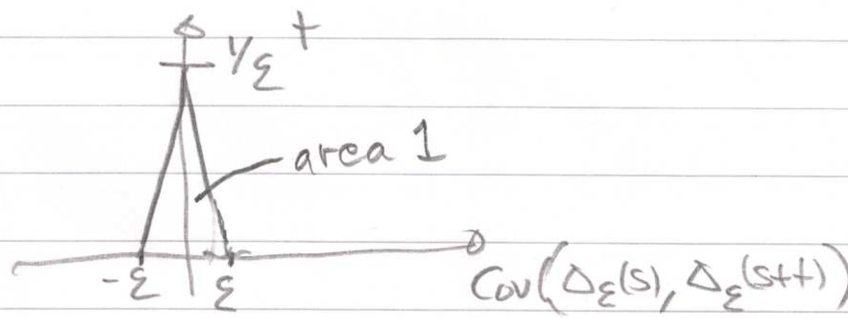
Solution

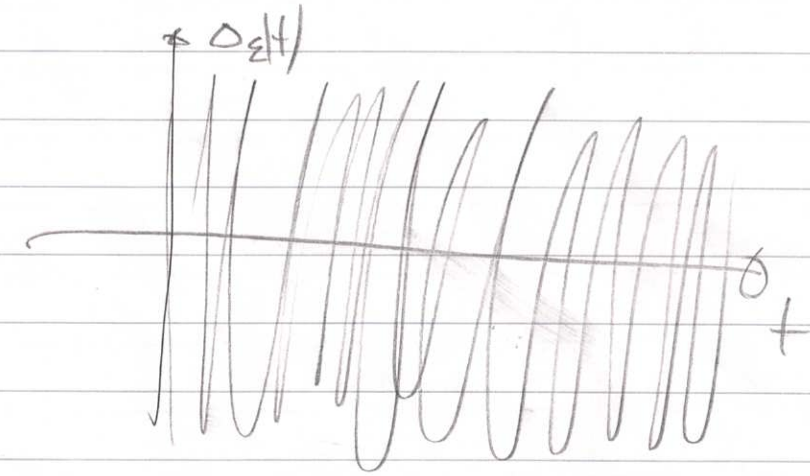
$$\mathrm{Cov}(\Delta_\varepsilon(s), \Delta_\varepsilon(s+t)) = E\left(\frac{B(s+\varepsilon)-B(s)}{\varepsilon} \, \frac{B(s+t+\varepsilon)-B(s+t)}{\varepsilon} \right)$$

$$= \frac{\min(s+\varepsilon, s+t+\varepsilon) - \min(s+\varepsilon, s+t) - \min(s, s+t+\varepsilon) + \min(s, s+t)}{\varepsilon^2}$$

$$= \frac{1}{\varepsilon^2}\left(\min(\varepsilon, t+\varepsilon) - \min(\varepsilon, t) - \min(0, t+\varepsilon) + \min(0,t) \right)$$

$$= \frac{1}{\varepsilon^2} \begin{cases} \varepsilon - \varepsilon - 0 - 0 = 0 & \text{for } t \geq \varepsilon \\ \varepsilon - t - 0 - 0 = \varepsilon - t & \text{for } 0 \leq t \leq \varepsilon \\ \varepsilon + t - t - 0 - t = \varepsilon + t & \text{for } -\varepsilon \leq t \leq 0 \\ t+\varepsilon - t - (t+\varepsilon) - t = 0 & \text{for } t \leq -\varepsilon \end{cases}$$





$\mathrm{Cov}(\Delta_\varepsilon(s), \Delta_\varepsilon(s+t)) \to \delta(t)$ as $\varepsilon \downarrow 0$ which is what it should be for white noise.