Fourier-transform/continuous time

1-dim Fouriertransform of $f:\mathbb{R}\to\mathbb{C}$

$$\hat{F}(\omega) = (Ff)(\omega) = \int_{-\infty}^{+\infty} e^{-j\omega x} f(x)\,dx \qquad (j^2=-1)$$

1-dim inverse Fourier transform of $\hat{f}:\mathbb{R}\to\mathbb{C}$

$$f(x) = (F^{-1}\hat{f})(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\omega x}\,\hat{F}(\omega)\,d\omega$$

n-dim Fouriertransform of $f:\mathbb{R}^n\to\mathbb{C}$

$$\hat{f}(\omega_1,\dots,\omega_n) = (Ff)(\bar{\omega}) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} e^{-j(\omega_1x_1+\dots+\omega_nx_n)} f(x_1,\dots,x_n)\,dx_1\dots dx_n$$

n-dim inverse Fouriertransform of $\hat{f}:\mathbb{R}^n\to\mathbb{C}$

$$f(x_1,\dots,x_n) = (F^{-1}\hat{f})(x_1,\dots,x_n) = \frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} e^{j(\omega_1x_1+\dots+\omega_nx_n)}\hat{f}(\omega_1,\dots,\omega_n)\,d\omega_1\dots d\omega_n$$

<u>example</u> for $f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}$ we have

This square is missing in movie by mistake

$$\hat{f}(\omega) = \underbrace{\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu+j\omega\sigma^2)^2/2\sigma^2}dx}_{\text{integral of } N(\cdot,\cdot)\text{-PDF which is 1}}\; e^{-j\omega\mu+\frac{1}{2}\omega^2\sigma^2} = e^{-j\omega\mu-\frac{1}{2}\omega^2\sigma^2} \quad \#$$

<u>example</u> for $f(x) = \begin{cases}\lambda e^{-\lambda x} & \text{for } x\geq 0\\ 0 & \text{for } x<0\end{cases}$ we have

$$\hat{f}(\omega) = \int_0^{+\infty}\lambda e^{-(\lambda+j\omega)x}dx = \frac{\lambda}{\lambda+j\omega} \quad \#$$

<u>1-dim properties</u>

$$F(f(x-x_0))(\omega) = e^{-j\omega x_0}(Ff)(\omega)$$
$$F(e^{j\omega_0 x}f(x))(\omega) = (Ff)(\omega-\omega_0)$$
$$F(f(-x))(\omega) = (Ff)(-\omega)$$
$$F(f'(x))(\omega) = j\omega(Ff)(\omega)$$

Fourier-transform / discrete time

1-dim Fourier transform of $f : \mathbb{Z} \to \mathbb{C}$

$$\hat{f}(\omega) = (F f)(\omega) = \sum_{k=-\infty}^{+\infty} e^{-j\omega k} f(k)$$

1-dim inverse Fourier transform of $\hat{f} : \mathbb{R} \to \mathbb{C}$

$$f(k) = (F^{-1} \hat{f})(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega k} \hat{f}(\omega)\, d\omega$$

<u>properties</u>

$$F(f(k-k_0))(\omega) = e^{-j\omega k_0} (F f)(\omega)$$
$$F(e^{j\omega_0 k} f(k))(\omega) = (F f)(\omega - \omega_0)$$
$$F(f(-k))(\omega) = (F f)(-\omega)$$

<u>example</u> for $f(k) = \frac{\lambda^k}{k!} e^{-\lambda}$ for $k = 0, 1, 2, \ldots$ (0 otherwise)

$$\hat{f}(\omega) = \underbrace{\sum_{k=0}^{+\infty} (\lambda e^{-j\omega})^k \frac{1}{k!}}_{\text{Taylor-exp of exp}} e^{-\lambda} = e^{\lambda(e^{-j\omega} - 1)} \quad \#$$

δ-function

* discrete time Kronecker δ-function $\delta : \mathbb{Z} \to \{0, 1\}$ $\delta(k) = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0 \end{cases}$

* continuous time Dirac δ-function $\delta : \mathbb{R} \to \{0, \infty\}$ given by

$$\int_{-\infty}^{+\infty} f(x)\delta(x)\, dx = f(0)$$ for $f : \mathbb{R} \to \mathbb{R}$ "smooth" and vanishing at infinity (with compact support)

$\delta(x)$ is the distribution derivative of the Heavyside step-function $\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$ as

$$\int_{-\infty}^{+\infty} f(x)\theta'(x)\, dx = \underbrace{\Big[f(x)\theta(x)\Big]_{-\infty}^{+\infty}}_{=0 \text{ since compact support}} - \int_{0}^{+\infty} f'(x)\, dx = f(0)$$

$(F\delta)(\omega) = 1$ (both in discrete and continuous time)

Convolution

continuous-time convolution between $f, g : \mathbb{R} \to \mathbb{R}$ is given by

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-y) g(y)\, dy = \int_{-\infty}^{+\infty} g(x-y) f(y)\, dy = (g * f)(x)$$

discrete time convolution between $f, g : \mathbb{Z} \to \mathbb{R}$ is given by

$$(f * g)(k) = \sum_{\ell=-\infty}^{+\infty} f(k-\ell) g(\ell) = \sum_{\ell=-\infty}^{+\infty} g(k-\ell) f(\ell) = (g * f)(k)$$

$$F(f * g)(\omega) = (Ff)(\omega)\,(Fg)(\omega) \text{ (both in discrete and continuous time)}$$

<u>example</u> for $f(x) = g(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$ we have

$$(f * g)(x) = \int_{-\infty}^{+\infty} \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(\sqrt{2}y - x/\sqrt{2})^2/2\sigma^2}}_{\text{integrates to } 1/\sqrt{2}} \, e^{-\frac{x^2}{4\sigma^2}}\, dy = \frac{1}{\sqrt{2\pi}\,\sqrt{2}\,\sigma} e^{-\frac{x^2}{2(\sqrt{2}\sigma)^2}}$$

when $\mu = 0$ and in a similar fashion

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}\,\sqrt{2}\,\sigma} e^{-\frac{(x-\mu)^2}{2(\sqrt{2}\sigma)^2}} \text{ for } \mu \text{ in general} \quad \#$$

Axioms of Probability Theory

S is the <u>sample space</u> of possible <u>outcomes</u> $\xi \in S$ of a random experiment, subsets of S (= collection of outcomes) are called <u>events</u>, a probability measure $P(A)$ is defined for events $A \subseteq S$ according to the axioms

$$P(\emptyset) = 0, \; P(S) = 1, \; P(A) \geq 0 \text{ and } P(A \cup B) = P(A) + P(B) \text{ for } A \cap B = \emptyset$$

From these axioms several additional rules follow:

$P(A^c) = P(\bar{A}) = 1 - P(A)$
$P(A) \leq P(B)$ for $A \subseteq B$
$P(A) \leq 1$
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$ when $A_i \cap A_j = \phi$ for $i \neq j$

Independence events A and B are independent if $P(A \cap B) = P(A)\,P(B)$

Conditional probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Law of total probability $P(B) = \sum_{i=1}^{n} P(B|A_i) P(A_i)$
whenever $A_i \cap A_j = \phi$ for $i \neq j$ and $\cup_{i=1}^{n} A_i = S$

Random variable function $X : S \to \mathbb{R}$ $X(\xi)$
function $(X, Y) : S \to \mathbb{R}^2$ $(X(\xi), Y(\xi))$

Cummulative Distribution Function (CDF) ~~$F_X(x) = P(X$~~
$F_X(x) = P(X \leq x) = P(\xi \in S : X(\xi) \leq x)$
$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$

Properties $0 \leq F_X(x) \leq 1$, $F_X(x_1) \leq F_X(x_2)$ for $x_1 \leq x_2$
$F_X(\infty) = 1$, $F_X(-\infty) = 0$, $P(a < X \leq b) = F_X(b) - F_X(a)$

$0 \leq F_{X,Y}(x,y) \leq 1$, $F_{X,Y}(x,y)$ is increasing in each argument
$F_{X,Y}(x, \infty) = F_X(x)$, $F_{X,Y}(\infty, y) = F_Y(y)$
$P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1) + F_{X,Y}(x_1,y_1)$

Independence X and Y independent if $P(X \in A, Y \in B)$
$= P(X \in A)\,P(Y \in B)$ for all $A, B \subseteq \mathbb{R}$

X and Y independent $\Leftrightarrow F_{X,Y}(x,y) = F_X(x) F_Y(y)$

* $\boxed{\text{Continuous random variable}}$ X or (X,Y) has uncountably infinitely many possible values, probability ~~mass~~ density function PDF

$$f_X(x) = \frac{d}{dx} F_X(x) \qquad f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

Properties $f_X(x) \geq 0$, $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$, $P(a < X \leq b) = \int_a^b f_X(x)\,dx$

$P(X \in A) = \int_A f_X(x)\,dx$

$f_{X,Y}(x,y) \geq 0$, $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1$, $P((X,Y) \in A) = \iint_A f_{X,Y}(x,y)\,dx\,dy$

$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy$, $f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx$

X and Y independent $\Leftrightarrow f_{X,Y}(x,y) = f_X(x) f_Y(y)$

example Gaussian $N(\mu, \sigma^2)$ r.v. $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$

example exponential r.v. with parameter λ $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

example uniform distribution over $[a,b]$ $f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$

Expectation $E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx$ $\quad E(g(X)) = \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx$

$E(g(X,Y)) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$

Conditional PDF of X given that $Y = y$

$f_{X|Y}(x|y) = f_{X,Y}(x,y) / f_Y(y)$

$P(X \in A | Y = y) = \int_A f_{X|Y}(x|y)\,dx \qquad P(X \in A) = \int_{-\infty}^{+\infty} P(X \in A | Y = y) f_Y(y)\,dy$

$E(X | Y = y) = \int_{-\infty}^{+\infty} x f_{X|Y}(x|y)\,dx \qquad E(X) = \int_{-\infty}^{+\infty} E(X | Y = y) f_Y(y)\,dy$

* $\boxed{\text{Discrete random variable}}$ X or (X,Y) has finitely or countably infinitely many possible values, probability mass function PMF

$$p_X(x) = P(X=x) \qquad p_{X,Y}(x,y) = P(X=x, Y=y)$$

Properties $p_X(x) \in [0,1]$, $\sum_{\text{all } x} p_X(x) = 1$, $P(X\in A) = \sum_{x\in A} p_X(x)$

$p_{X,Y}(x,y) \in [0,1]$, $\sum\sum_{\text{all } x,y} p_{X,Y}(x,y) = 1$, $P((X,Y)\in A) = \sum\sum_{(x,y)\in A} p_{X,Y}(x,y)$

$p_X(x) = \sum_{\text{all } y} p_{X,Y}(x,y)$, $p_Y(y) = \sum_{\text{all } x} p_{X,Y}(x,y)$

X and Y independent $\iff$ $p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$

example Bernoulli r.v. X, possible values $\{0,1\}$, $p_X(0) = 1-p$, $p_X(1) = p$

example Binomial r.v. X, possible values $\{0,\dots,n\}$, $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k=0,\dots,n$, same as the sum of n independent Bernoulli r.v.'s

example Poisson r.v. X, possible values $\{0,1,\dots\} = \mathbb{N}$, $p_X(k) = \lambda^k e^{-\lambda}/k!$ for $k=0,1,2,\dots$, $Po(\lambda_1) + Po(\lambda_2) = Po(\lambda_1+\lambda_2)$, that is sum of two independent Po is Po

example geometric (waiting time) ~~r.v.~~ X, possible values $\{1,2,3,\dots\}$, $p_X(k) = (1-p)^{k-1} p$ for $k=1,2,\dots$, number of Bernoulli trials that have to be made until first 1 occur...

Expectation $E(X) = \sum_{\text{all } x} x\, p_X(x)$ $\qquad E(g(X)) = \sum_{\text{all } x} g(x)\, p_X(x)$

$E(g(X,Y)) = \sum_{\text{all } x}\sum_{y} g(x,y)\, p_{X,Y}(x,y)$

Conditional PMF of X given that $Y=y$

$$p_{X|Y}(x|y) = p_{X,Y}(x,y) / p_Y(y)$$

$$P(X\in A | Y=y) = \sum_{x\in A} p_{X|Y}(x|y) \qquad P(X\in A) = \sum_{\text{all } y} P(X\in A|Y=y)\, p_Y(y)$$

$$E(X|Y=y) = \sum_{\text{all } x} x\, p_{X|Y}(x|y) \qquad E(X) = \sum_{\text{all } y} E(X|Y=y)\, p_Y(y)$$

Linearity of expectation $E\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n E(X_i)$

Variance $Var(X) = \sigma_X^2 = E\left((X-E(X))^2\right) = E(X^2) - (E(X))^2$

Covariance $Cov(X,Y) = E\left((X-E(X))(Y-E(Y))\right) = E(XY) - E(X)E(Y)$

$Var(X) = Cov(X,X)$

X and Y are called uncorrelated if $Cov(X,Y)=0$

X and Y independent $\Rightarrow$ X and Y uncorrelated

Bilinearity of covariance $Cov\left(\sum_{i=1}^m a_i X_i, \sum_{j=1}^n b_j Y_j\right) = \sum_{i=1}^m \sum_{j=1}^n a_i b_j Cov(X_i, Y_j)$

in particular $Var\left(\sum_{i=1}^n a_i X_i\right) = Cov\left(\sum_{i=1}^n a_i X_i, \sum_{j=1}^n a_j X_j\right) = \sum_{i=1}^n \sum_{j=1}^n a_i a_j Cov(X_i, X_j)$

Bivariate normal distribution

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2}{2(1-\rho^2)}\right)$$

n-variate normal distribution see page 111 in Hsu (page 89 in first edition)

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{\det K}} \exp\left(-\tfrac{1}{2}(x-\mu)^T K^{-1}(x-\mu)\right)$$ where

n×n-matrix

column matrices

$\mu_i = E(X_i)$ and $K_{ij} = Cov(X_i, X_j)$

Functions of random variables $\underline{Y} = g(\underline{X})$, $g: \mathbb{R} \to \mathbb{R}$ increasing

$\Rightarrow f_{\underline{Y}}(y) = \frac{d}{dy} P(\underline{Y} \le y) = \frac{d}{dy} P(\underline{X} \le g^{-1}(y)) = f_{\underline{X}}(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$

$\underline{Z} = g(\underline{X}, \underline{Y}) \quad f_{\underline{Z}}(z) = \frac{d}{dz} P(g(\underline{X}, \underline{Y}) \le z) = \frac{d}{dz} \iint_{g(x,y) \le z} f_{\underline{X},\underline{Y}}(x,y)\,dx\,dy$

Characteristic function CHF of continuous random variable(s)

$\varphi_{\underline{X}}(\omega) = E(e^{j\omega \underline{X}}) = \int_{-\infty}^{+\infty} e^{j\omega x} f_{\underline{X}}(x)\,dx \quad | \, f_{\underline{X}}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-j\omega x} \varphi_{\underline{X}}(\omega)\,d\omega$

$\varphi_{\underline{X},\underline{Y}}(\omega_1, \omega_2) = E(e^{j(\omega_1 \underline{X} + \omega_2 \underline{Y})}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{j(\omega_1 x + \omega_2 y)} f_{\underline{X},\underline{Y}}(x,y)\,dx\,dy$

$f_{\underline{X},\underline{Y}}(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{(2\pi)^2} e^{-j(\omega_1 x + \omega_2 y)} \varphi_{\underline{X},\underline{Y}}(\omega_1, \omega_2)\,d\omega_1\,d\omega_2$