# Grimmett and Stirzaker 6.1-6.4

**Def** A time discrete and value discrete random process $\{X_n\}_{n=0}^{+\infty}$ satisfies the Markov condition if

$$P(X_{n+1}=s_{n+1} \mid X_n=s_n,\dots,X_0=s_0) = P(X_{n+1}=s_{n+1} \mid X_n=s_n)$$

for $s_0,\dots,s_{n+1} \in S$ in the state space of possible values of the process and $n \in \mathbb{N}$.

* Markov condition turns out to be equivalent with

$$P(X_{n_{k+1}}=s_{k+1} \mid X_{n_k}=s_k,\dots,X_{n_0}=s_0) = P(X_{n_{k+1}}=s_{k+1} \mid X_{n_k}=s_k)$$

for $s_0,\dots,s_{k+1} \in S$ and $0 \leq n_0 < \dots < n_{k+1}$ as well as

$$P(X_{m+n}=s_{m+n} \mid X_n=s_n,\dots,X_0=s_0) = P(X_{m+n}=s_{m+n} \mid X_n=s_n)$$

for $s_0,\dots,s_n, s_{m+n} \in S$, $n \in \mathbb{N}$ and $m \geq 1$.

* As $S$ is assumed discrete it is no restriction assume that $S \subseteq \mathbb{Z}$ as values otherwise can be recoded to satisfy this.

**Def** A process $\{X_n\}_{n=0}^{+\infty}$ satisfying the Markov condition is called a Markov chain.

**Def** A Markov chain is (time) homogeneous if

$p_{ij} = P(X_{n+1}=j \mid X_n=i) = P(X_1=j \mid X_0=i)$ do not depend on $n$

The transition matrix $P$ is the matrix with elements $(P)_{ij} = p_{ij}$ the transition probabilities.

* All theory we develop will be for time homogeneous Markov chains, but we will see some examples of non-homogeneous chains in exercises.

**Thm** The transition matrix is a stochastic matrix which is to say that it has non-negative elements with row sums 1.

**Proof** Immediate! #

**Def** The $n$-step transition matrix $P(m, m+n)$ is has elements $(P(m,m+n))_{ij} = P(X_{m+n}=j \mid X_m=i)$ $= p_{ij}(m, m+n)$ the $n$-step transition probabilities.

**Thm** $P(m, m+n) = P^n$ for $m \geq 0$ and $n \geq 1$

**Proof** True by definition for $n=1$ while for $n>1$

$$(P(m,m+n))_{ij} = P(X(m+n)=j \mid X_m=i) = \sum_k P(X(m+n)=j, X(m+n-1)=k \mid X_m=i)$$

$$= \sum_k \frac{P(X(m+n)=j, X(m+n-1)=k, X_m=i)}{P(X(m+n-1)=k, X_m=i)} \frac{P(X(m+n-1)=k, X_m=i)}{P(X_m=i)}$$

$$= \sum_k (P)_{kj} P(m, m+n-1)_{ik} = (P(m,m+n-1)P)_{ij} = \ldots = (P^n)_{ij} \#$$

**Def** $\boxed{\text{The } \underline{\text{distribution row matrix}}\ \mu^{(n)} \text{ of } X_n \text{ has entries } \mu^{(n)}_j = P(X_n = j)}$

**Thm** $\boxed{\mu^{(m+n)} = \mu^{(m)} P^n \text{ and in particular } \mu^{(n)} = \mu^{(0)} P^n}$

**Proof** $\mu^{(m+n)}_j = P(X_{m+n}=j) = \sum_k P(X_{m+n}=j \mid X_m = k) P(X_m = k)$

$$= \sum_k \mu^{(m)}_k (P^n)_{kj} = (\mu^{(m)} P^n)_j \quad \#$$

* As we have shown that $p(m, m+n)_{ij}$
$= P(X_{m+n}=j \mid X_m = i) = P(X_n = j \mid X_0 = i) = (P^n)_{ij}$
does not depend on $m$ we denote this
n-step transition probability $p_{ij}(n)$

Example (simple random walk) $S = \mathbb{Z}$ and

$$p_{ij} = \begin{cases} p, & j = i+1 \\ q = 1-p, & j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

$$p_{ij}(n) = \begin{cases} \binom{n}{\frac{1}{2}(n+j-i)} p^{\frac{1}{2}(n+j-i)} q^{\frac{1}{2}(n-j+i)}, & n+j-i \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

since $\begin{cases} \text{steps up} - \text{steps down} = j-i \\ \text{steps up} + \text{steps down} = n \end{cases} \Leftrightarrow \begin{cases} \text{steps up} = \frac{1}{2}(n+j-i) \\ \text{steps down} = \frac{1}{2}(n-j+i) \end{cases}$ #

# Classification of States

**Def** State (value) $i$ is reccurent = persistent if

$$P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i) = 1.$$

otherwise $i$ is transient.

**Def**

$$f_{ij}(n) = \begin{cases} P(X_n = j, X_{n-1} \neq j, \dots, X_1 \neq j \mid X_0 = i) & n \geq 1 \\ 0 & n = 0 \end{cases}$$

$$f_{ij} = \sum_{n=0}^{+\infty} f_{ij}(n) = \sum_{n=1}^{+\infty} f_{ij}(n) \qquad p_{ij}(0) = \delta_{ij}$$

$$P_{ij}(s) = \sum_{n=0}^{+\infty} s^n p_{ij}(n) \qquad F_{ij}(n) = \sum_{n=0}^{+\infty} s^n f_{ij}(n) \quad |s| < 1$$

* Note that $i$ is persisten if and only if $f_{ii} = 1$.

**Thm**

$$P_{ii}(s) = 1 + F_{ii}(s) P_{ii}(s)$$

$$P_{ij}(s) = F_{ij}(s) P_{jj}(s) \quad \text{for } i \neq j$$

**Proof**

$$P_{ii}(s) = \sum_{n=0}^{+\infty} s^n p_{ii}^{(n)} = 1 + \sum_{n=1}^{+\infty} \sum_{k=1}^{n} f_{ii}(k) p_{ii}(n-k) s^n$$

$$= 1 + \sum_{k=1}^{+\infty} f_{ii}(k) s^k \sum_{n=k} s^{n-k} p_{ii}(n-k) = 1 + F_{ii}(s) P_{ii}(s)$$

$$P_{ij}(s) = \sum_{n=1}^{+\infty} s^n p_{ij}(n) = \sum_{n=1}^{+\infty} \sum_{k=1}^{n} f_{ij}(k) p_{jj}(n-k) s^n = F_{ij}(s) \overset{i \neq j}{P_{jj}(s)} \#$$

**Thm** ① $j$ is persistent if and only if $\sum_n p_{jj}(n) = \infty$ and then $\sum_n p_{ij}(n) = \infty$ when $f_{ij} > 0$

② $j$ is transient if and only if $\sum_n p_{jj}(n) < \infty$ and then $\sum_n p_{ij}(n) < \infty$

**Proof** ① As $F_{jj}(s) = \sum_{n=1}^{+\infty} s^n f_{jj}(n) \nearrow \sum_{n=1}^{+\infty} f_{jj}(n) = f_{jj}$ and $P_{jj}(s) = \sum_{n=0}^{+\infty} s^n p_{jj}(n) \nearrow \sum_{n=0}^{+\infty} p_{jj}(n)$ as $s \uparrow 1$

$$\infty = \sum_{n=0}^{+\infty} p_{jj}(n) = \lim_{s\uparrow 1} P_{jj}(s) = \lim_{s \uparrow 1} \frac{1}{1 - F_{jj}(s)} \Leftrightarrow f_{jj} = 1$$

② By ① $f_{jj} < 1 \Leftrightarrow \sum_{n=0}^{+\infty} p_{jj}(n) < \infty$ and then

$$\sum_{n=0}^{+\infty} p_{ij}(n) = \lim_{s \uparrow 1} \sum_{n=0}^{+\infty} s^n p_{ij}(n) = \lim_{s\uparrow 1} P_{ij}(s) = \lim_{s \uparrow 1} F_{ij}(s) P_{jj}(s) < \infty$$

as $\lim_{s\uparrow 1} F_{ij}(s) = \sum_{i=1} f_{ij}(n) \leq 1$. #

**Thm** $\boxed{j \text{ transient} \Rightarrow \lim_{n\to\infty} p_{jj}(n) = 0}$

**Proof** Immediate from previous! #

**Def** With $T_j = \min\{n \geq 1 : X_n = j\}$ the <u>mean reccurence time</u> $\mu_j$ of state $j$ is

$$\mu_j = E(T_j \mid X_0 = j) = \begin{cases} \sum_{n=1}^{+\infty} n f_{jj}(n) & \text{for } f_{jj} = 1 \\ +\infty & \text{for } f_{jj} < 1 \end{cases}$$

* Note that $\mu_j = \infty$ is possible also for $f_{jj} = 1$.

**Def** A persistent state $j$ is called
<u>null</u> if $\mu_j = \infty$ and <u>non-null</u> if $\mu_j < \infty$

**Deep Thm** A persistent state $j$ is null $\Leftrightarrow \lim_{n\to\infty} p_{jj}(n) = 0$
In that case $\lim_{n\to\infty} p_{ij}(n) = 0$ for all $i$.

**Def** The <u>period</u> of state $j$ is $d(j) = \gcd\{n \geq 1 : p_{jj}(n) > 0\}$
A state is called <u>aperiodic</u> if it has period 1.

**Def** A state is called <u>ergodic</u> if non-null persistent and aperiodic.

## Classification of chains

**Def**
* <u>$i$ communicates with $j$, $i \to j$</u>, if $p_{ij}(n) > 0$ for some $n \geq 0$
* <u>$i$ and $j$ intercommunicate</u>, $i \leftrightarrow j$, if $i \to j$ & $j \to i$

* It is easy to see that $\leftrightarrow$ is an equivalence relation.

**Thm** If $i \leftrightarrow j$ then $i$ and $j$ has same period, $i$ is transient iff $j$ is and $i$ is non-null persistent iff $j$ is.

Partial **Proof** As $i \leftrightarrow j$ we have $p_{ij}(m)\, p_{ji}(n) > 0$, some $m, n \geq 0$ so that $p_{ii}(m+n+r) \geq p_{ij}(m)\, p_{jj}(r)\, p_{ji}(n)$ giving $\sum_r p_{jj}(r) < \infty$ if $\sum_r p_{ii}(r) < \infty$, i.e., $i$ transient $\Rightarrow$ $j$ transient. #

**Def** A set $C$ of states is called
* <u>closed</u> if $p_{ij} = 0$ for $i \in C$ and $j \notin C$
* <u>irreducible</u> if $i \leftrightarrow j$ for $i, j \in C$

**Thm** The state space can be decomposed as $S = T \cup C_1 \cup C_2 \cup \ldots$ where $T$ are the transient statets and $C_1, C_2$ are closed irreducible sets of persistent states. The decomposition is unique except for the order betwee the $C_i$'s.

**Proof** Let $T$ be the transient states and $C_1, C_2, \ldots$ the equivalence classes of $\leftrightarrow$ for the remaining persistent states. To prove $C_k$ is closed assume the contrary so that $p_{ij} > 0$ for some $i \in C_k$ and $j \notin C_k$. We do not have $j \to i$ as well since then $j$ would be in $C_k$. But this means $i$ is not persistent which is a contradiction. #

**Thm** If $S$ is finite then at least one state is persistent and all persistent states are non-null.

**Proof** $1 = \sum_{j \in S} p_{ij}(n) \to 0$ as $n \to \infty$ if all $j$ are transient as in that case $\sum_n p_{ij}(n) < \infty$ so that $p_{ij}(n) \to 0$ as $n \to \infty$ for all $j$. Consider the closed set $C$ of all null persistent states. Then

$1 = \sum_{j \in C} p_{ij}(n) \to 0$ as $n \to \infty$ wich is again wrong! #

# Stationary distribution and the limit theorem

**Def** A row matrix $\pi$ with positive elements adding up to 1 is a stationary distribution if $\pi = \pi P$.

**Thm** If $\pi$ is stationary distribution then $\pi P^n = \pi$

**Proof** Immediate! #

**Thm** An irreducible chain has a stationary distribution $\pi$ if and only if all states are non-null persistent and in that case $\pi_i = 1/\mu_i$

**Thm** For any aperiodic $j$

$$p_{jj}(n) \to 1/\mu_j \text{ as } n \to \infty$$

$$p_{ij}(n) = f_{ij}/\mu_j \text{ as } n \to \infty \text{ for } i \neq j$$