$\boxed{6.1.1}$ $P(X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0)$

$= P(X_{n+1}=j) = P(X_{n+1}=j \mid X_n=i)$

time homogeneous iff sequence is iid

$\boxed{6.1.2}$ (a) $P(X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \ldots) = \begin{cases} 0 & \text{for } j<i \\ i/6 & \text{for } j=i \\ 1/6 & \text{for } j>i \end{cases}$

(b) $P(N_{n+1}=j \mid N_n=i, N_{n-1}=i_{n-1}, \ldots) = \begin{cases} 5/6 & \text{for } j=i \\ 1/6 & \text{for } j=i+1 \\ 0 & \text{otherwise} \end{cases}$

(c) $P(C_{n+1}=j \mid C_n=i, C_{n-1}=i_{n-1}, \ldots) = \begin{cases} 5/6 & \text{for } j=i+1 \\ 1/6 & \text{for } j=0 \\ 0 & \text{otherwise} \end{cases}$

(d) $P(B_{n+1}=j \mid B_n=i, B_{n-1}=i_{n-1}, \ldots) = \begin{cases} 1 & \text{for } j=i-1\geq 0 \\ \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{j-1} & \text{for } \substack{i=0 \\ j\geq 1} \\ 0 & \text{otherwise} \end{cases}$

$B_{n+1} = \begin{cases} B_n - 1 & \text{for } B_n>0 \\ \underbrace{Y}_{\text{waiting time}} & \text{for } B_n=0 \end{cases}$

$\boxed{6.1.4}$ (a) $P(Y_{r+1}=i_{r+1} \mid Y_r=i_r, \ldots, Y_0=i_0) = P(X_{n_{r+1}}=i_{r+1} \mid X_{n_r}=i_r, \ldots)$

$= p_{i_r i_{r+1}}(n_{r+1}-n_r)$

$P(X_{2n+2}=j \mid X_{2n}=i) = \begin{cases} 2p(1-p) & j=i \\ p^2 & j=i+2 \\ (1-p)^2 & j=i-2 \end{cases}$

6.1.10 $P(X_n = k \mid X_0 = x_0, \dots, X_{n-1} = x_{n-1}, X_{n+1} = x_{n+1}, \dots, X_N = x_N)$

$$= \frac{P(X_n = k, X_0 = x_0 \text{ ——— '' ——— })}{P(X_0 = x_0, \text{ ——— '' ——— })}$$

$$= \frac{p_{x_0 x_1} \cdots p_{x_{n-1} k}\, p_{k x_{n+1}} \cdots p_{x_{N-1} x_N}}{p_{x_0 x_1} \cdots p_{x_{n-1} x_{n+1}}(2) \cdots p_{x_{N-1} x_N}} \times \frac{P(X_{n-1} = x_{n-1})}{P(X_{n-1} = x_{n-1})}$$

$$= \frac{P(X_{n+1} = x_{n+1}, X_n = k, X_{n-1} = x_{n-1})}{P(X_{n+1} = x_{n+1}, X_{n-1} = x_{n-1})}$$

$$= P(X_n = k \mid X_{n+1} = x_{n+1}, X_{n-1} = x_{n-1})$$

6.1.12 $\sum_i (P^n)_{ij} \overset{=}{\leq} 1 \quad n = 1, \dots, N$ implies

$$\sum_i (P^{N+1})_{ij} = \sum_i \sum_k (P^N)_{ik} P_{kj}$$

$$= \sum_k P_{kj} \sum_i (P^N)_{ik} \overset{=}{\leq} \sum_k P_{kj} \overset{=}{\leq} 1$$

6.2.1 $l_{ij}(n) = P(X_n = j, X_k \neq i \text{ for } 1 \leq k < n, X_0 = i)$

$$L_{ij}(s) = \sum_{n=1}^{+\infty} s^n l_{ij}(n)$$

$$P_{ij}(s) = \sum_{n=1}^{+\infty} s^n p_{ij}(n) = \sum_{n=1}^{+\infty} \sum_{k=0}^{n-1} s^n p_{ii}(k)\, l_{ij}(n-k)$$

$$= \sum_{k=0}^{+\infty} s^k p_{ii}(k) \sum_{n=k+1}^{+\infty} s^{n-k} l_{ij}(n-k) = P_{ii}(s) L_{ij}(s)$$

$$= F_{ij}(s) P_{jj}(s)$$

So $F_{ij}(s) = L_{ij}(s)$ if $P_{ii}(s) = P_{jj}(s)$ as, e.g., for simple walk.

6.2.2 $P(\text{no return to } i \mid X_0=i) \geq \underbrace{P(X_{n_i}=s \mid X_0=i)}_{>0}$ for

$n_i = \min\{n \geq 1 : p_{is}(n) > 0\}$

as X_n chain makes no visit to i for $n \in [1, n_i - 1]$

6.2.3 Lettin $I_k(\omega) = \begin{cases} 1 & \text{if } X_k(\omega)=i \\ 0 & \text{if } X_k(\omega) \neq i \end{cases}$ the number of visits N to i is $\sum_{k=0}^{+\infty} I_k$. It follows that $E(N \mid X_0=i) = \sum_{k=0}^{+\infty} p_{ii}(k) = \infty$ iff i is ~~persistent~~ reccurent.

6.3.2 $\begin{cases} p_{i,i+2} = p \\ p_{i,i-1} = 1-p \end{cases}$ the mean jump-size is $3p-1$ and it follows that we have persistence iff $p = 1/3$

6.3.2

Alternatively

$$p_{ii}(n) = \begin{cases} p^k (1-p)^{2k} \binom{3k}{k} & \text{for } n = 3k \\ 0 & \text{for } n \neq 3k \end{cases}$$

and then proceed as we studied this problem for ordinary simple random walk $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$

6.3.3 (a) $p_{ii}(n)$ I do not have the energy for ....

$p > 0$

$$\begin{cases} \pi_1 (1-2p) + \pi_2 p = \pi_1 \iff \pi_2 p = 2p\,\pi_1 \iff \pi_2 = 2\pi_1 \\ \pi_1 2p + \pi_2 (1-2p) + \pi_3 2p = \pi_2 \iff \ldots \iff \pi_2 = 2\pi_3 \\ p\pi_2 + (1-2p)\pi_3 = \pi_3 \iff \pi_2 = 2\pi_3 \end{cases}$$

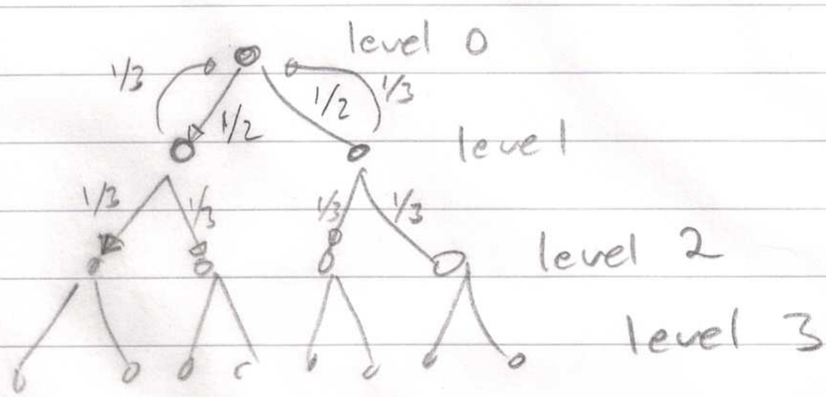
$$(\pi_1\ \pi_2\ \pi_3) = \left(\tfrac{1}{4}\ \tfrac{1}{2}\ \tfrac{1}{4}\right) = \left(\tfrac{1}{\mu_1}\ \tfrac{1}{\mu_2}\ \tfrac{1}{\mu_3}\right)$$

6.3.4 See page 6-7.

6.4.4 $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has any π as stationary

6.4.6 d_v = number of connections to vertex v
clearly π_v must be proportional to d_v,
as $\sum_v d_v = 2g$ we get $\pi_v = d_v/(2g)$

6.4.7



if X_n is level at time n we see that

$$P_{ij} = \begin{cases} 2/3 & \text{for } j = i+1 \\ 1/3 & \text{for } j = i-1 \end{cases}$$ so transient according to earlier

6.4.8 $X_{n+1} = \sum_{i=1}^{X_n} B_{i,n} + Y_n$ gives Markov as independent

of $X_0, \dots, X_{n-1}$. In equilibrium ~~we~~ we have that

$$G_{n+1}(s) = E(s^{X_{n+1}}) = E(s^{Po(\lambda)}) E(G_{Bin(1,q)}(s)^{X_n})$$

$$= \left(\sum_{n=0}^{+\infty} s^n \frac{\lambda^n}{n!} e^{-\lambda}\right) E((p+qs)^{X_n}) = e^{\lambda(s-1)} G_n(p+qs)$$

$$= G_n(s)$$

with solution $G_n(s) = e^{\lambda(s-1)/p}$ as

$$e^{\lambda(s-1)} e^{\lambda(p+qs-1)/p} = e^{\lambda s + \lambda q s/p - \lambda p} = e^{\lambda s + \lambda \frac{1-p}{p} s - \lambda/p}$$

$$= \dots\dots = e^{\lambda(s-1)/p}$$

6.5.1 The stationary distribution is $\pi_n = \pi_0 \frac{\lambda_0 \cdot \ldots \cdot \lambda_{n-1}}{\mu_1 \cdot \ldots \cdot \mu_n}$ and the equations to check $\pi_i p_{ij} = \pi_j p_{ji}$ boil down to $\pi_n p_{n,n+1} = \pi_{n+1} p_{n+1,n}$ which in turn obviously holds.

6.5.2 $\boxed{\Rightarrow}$ $\pi_{j_1} p_{j_1 j_2} p_{j_2 j_3} \cdot \ldots \cdot p_{j_{n-1} j_n} p_{j_n j_1} = p_{j_2 j_1} \pi_{j_2} p_{j_2 j_3} \cdot \ldots \cdot p_{j_{n-1} j_n} p_{j_n j_1}$
$= p_{j_2 j_1} p_{j_3 j_2} \cdot \ldots \cdot p_{j_n j_{n-1}} p_{j_1 j_n} \pi_{j_1}$

$\boxed{\Leftarrow}$ $\pi_i p_{ij} = \lim_{n\to\infty} p_{ji}(n) p_{ij} = p_{ij} \lim_{n\to\infty} \sum_{i_1 \ldots i_n} p_{j i_1} \cdot \ldots \cdot p_{i_n i}$
$= p_{ji} \lim_{n\to\infty} \sum_{i_1 \ldots i_n} p_{i i_n} \cdot \ldots \cdot p_{i_1 j} = p_{ji} \lim_{n\to\infty} p_{ij}(n) = \pi_j p_{ji}$

6.5.6 $\boxed{a}$ As $\pi = \left(\frac{\beta}{\alpha+\beta} \; \frac{\alpha}{\alpha+\beta}\right)$ equation $\pi_i p_{ij} = \pi_j p_{ji}$ boils down to $\alpha \frac{\beta}{\alpha+\beta} = \beta \frac{\alpha}{\alpha+\beta}$ which checks.

$\boxed{b}$ As $\pi = \left(\frac{1}{3} \; \frac{1}{3} \; \frac{1}{3}\right)$ equations $\pi_i p_{ij} = \pi_j p_{ji}$ hold iff $p = \frac{1}{2}$.

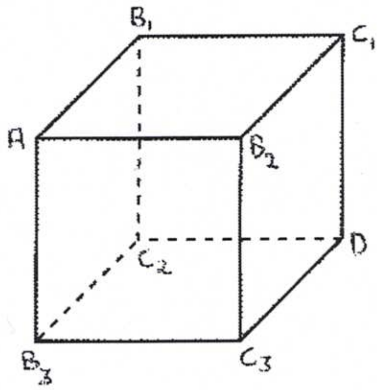
Exercise 6.3.4 in Grimmett and Stirzaker

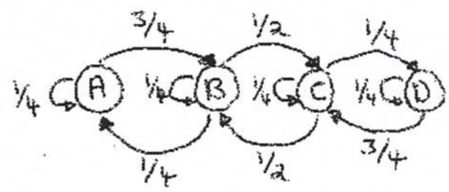
Task. A particle performs a discrete time random walk on the vertices of a cube. At each step it remains where it is with probability 1/4 or moves to one of its neighbouring vertices each having probability 1/4. Let A and D denote two diametrically opposite vertices. If the walk starts at A, find

(a) the mean number of steps until its first visit to D,

(b) the mean number of steps until its first return to A, and

(c) the mean number of visits to D before its first return to A.





Solution. (a) Let B_1, B_2 and B_3 denote the three vertices that are closest to A (= one step away from A) and C_1, C_2 and C_3 the three vertices that are closest to D (= one step away from D = two steps away from A). Introduce a four state Markov chain with values A, B, C and D indicating if the random walk is in A, in one of the states B_1, B_2 or B_3, in one of the states C_1, C_2 and C_3, or in the state D, respectively, with corresponding probability transition matrix

$$P = \begin{bmatrix} 1/4 & 3/4 & 0 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 0 & 1/2 & 1/4 & 1/4 \\ 0 & 0 & 3/4 & 1/4 \end{bmatrix}.$$

Writing T_{AD}, T_{BD} and T_{CD} for the mean number of steps until the first visit to D starting at A, B and C, respectively, we then have the following system of equations

$$\begin{cases} \mathbf{E}\{T_{\mathrm{AD}}\} = 1 + (1/4)\cdot\mathbf{E}\{T_{\mathrm{AD}}\} + (3/4)\cdot\mathbf{E}\{T_{\mathrm{BD}}\} \\ \mathbf{E}\{T_{\mathrm{BD}}\} = 1 + (1/4)\cdot\mathbf{E}\{T_{\mathrm{AD}}\} + (1/4)\cdot\mathbf{E}\{T_{\mathrm{BD}}\} + (1/2)\cdot\mathbf{E}\{T_{\mathrm{CD}}\} \\ \mathbf{E}\{T_{\mathrm{CD}}\} = 1 + (1/2)\cdot\mathbf{E}\{T_{\mathrm{BD}}\} + (1/4)\cdot\mathbf{E}\{T_{\mathrm{CD}}\} + (1/4)\cdot 0 \end{cases},$$

which in turn is obtained by conditioning on where we end up after one step on our journey to D starting at A, B and C, respectively. Solving this

```
In[1]:= Solve[{AD, BD, CD} == {1+AD/4+3*BD/4,
                    1+AD/4+BD/4+CD/2, 1+BD/2+CD/4}, {AD, BD, CD}]
Out[1]= {AD -> 40/3, BD -> 12, CD -> 28/3}
```

we arrive at the answer $\mathbf{E}\{T_{\mathrm{AD}}\} = 40/3$.

(b) Writing T_{AA} and T_{BA} for the mean number of steps until the next visit to A starting at A and B, respectively, we may use the result of task (a) together with some obvious symmetry properties to obtain

$$\mathbf{E}\{T_{\mathrm{AA}}\} = 1 + (1/4)\cdot 0 + (3/4)\cdot \mathbf{E}\{T_{\mathrm{BA}}\} = 1 + (3/4)\cdot \mathbf{E}\{T_{\mathrm{CD}}\} = 1 + 28/4 = 8.$$

(c) Write D_{AA}, D_{BA}, D_{CA} and D_{DA} for the mean number of visits to D before next visit to A when starting at A, B, C and D, respectively. In the fashion of the solution to task (a) we then have

$$\begin{cases} \mathbf{E}\{D_{\mathrm{AA}}\} = (1/4)\cdot 0 + (3/4)\cdot \mathbf{E}\{D_{\mathrm{BA}}\} \\ \mathbf{E}\{D_{\mathrm{BA}}\} = (1/4)\cdot 0 + (1/4)\cdot \mathbf{E}\{D_{\mathrm{BA}}\} + (1/2)\cdot \mathbf{E}\{D_{\mathrm{CA}}\} \\ \mathbf{E}\{D_{\mathrm{CA}}\} = (1/2)\cdot \mathbf{E}\{D_{\mathrm{BA}}\} + (1/4)\cdot \mathbf{E}\{D_{\mathrm{CA}}\} + (1/4)\cdot (\mathbf{E}\{D_{\mathrm{DA}}\}+1) \\ \mathbf{E}\{D_{\mathrm{DA}}\} = (3/4)\cdot \mathbf{E}\{D_{\mathrm{CA}}\} + (1/4)\cdot (\mathbf{E}\{D_{\mathrm{DA}}\}+1) \end{cases}$$

(remember that we now are counting the number of visits to D, not time, so we should not add time units on the right-hand side, but instead possible visits to D), with solution

```
In[2]:= Solve[{AA, BA, CA, DA} == {3*BA/4, BA/4+CA/2,
                  BA/2+CA/4+(DA+1)/4, 3*CA/4+(DA+1)/4}, {AA, BA, CA, DA}]
Out[2]= {AA -> 1, BA -> 4/3, CA -> 2, DA -> 7/3}
```

giving us the answer $\mathbf{E}\{D_{\mathrm{AA}}\} = 1$.