G&S 6.5 Reversibility

$X_n, n \geq 0$ irreducible non-null persisten in steady state π

$Y_n, n=0,\dots,N$ is X_{N-n}

$P(Y_{n+1}=j \mid Y_n=i, Y_{n-1}=y_{n-1},\dots,Y_0=y_0)$

$$= \frac{P(X_{N-n-1}=j, X_{N-n}=i, X_{N-n+1}=y_{n-1},\dots,X_N=y_0)}{P(X_{N-n}=i, X_{N-n+1}=y_{n-1},\dots,X_N=y_0)}$$

$$= \frac{\pi_j p^X_{ji} p^X_{i y_{n-1}} \cdot \dots \cdot p^X_{y_1 y_0}}{\pi_i p^X_{i y_{n-1}} \cdot \dots \cdot p^X_{y_1 y_0}} = \frac{\pi_j p^X_{ji}}{\pi_i}$$

does not depend on $y_{n-1},\dots,y_0$ so Y_n Markov with $p^Y_{ij} = \pi_j p^X_{ji}/\pi_i$

Def X_n reversible if $p^Y_{ij} = p^X_{ij} \Leftrightarrow \pi_i p^X_{ij} = \pi_j p^X_{ji}$

Thm If π is distribution row matrix such that $\pi_i p^X_{ij} = \pi_j p^X_{ji}$ then π is stationary distribution and X_n is reversible.

Proof By what we just did it is enough prove π stationary which follows from

$$\sum_i \pi_i p_{ij} = \sum_i \pi_j p_{ji} = \pi_j \quad \#$$

G&S 6.9 Continuous Markov chain

Although 6.8 and 6.11 are included in course they are special cases of 6.9 and need not be read except for that some exercises use some theorems from 6.8 and 6.11 so one must look up the theorem to do exercises.

$X(t), t \geq 0$ continuous time discrete valued process

Markov definition is as before with continuous time

$$p_{ij}(t) = P(X(t+s)=j \mid X(s)=i) = P(X(t)=j \mid X(0)=i)$$

$$(P_t)_{ij} = p_{ij}(t)$$

$$P_s P_t = P_{s+t} \text{ and } \mu^{(s+t)} = \mu^{(s)} P_t \text{ as before}$$

Def $P_0' \equiv \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(P_\varepsilon - I) = G$ the generator

Thm $P_t' = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(P_{t+\varepsilon} - P_t) = \begin{cases} P_t \lim_{\varepsilon \downarrow 0}(P_\varepsilon - I) = P_t G \\ = \lim_{\varepsilon \downarrow 0}(P_\varepsilon - I) P_t = G P_t \end{cases}$

Thm $P_t = e^{tG} = \sum_{k=0}^{+\infty} \frac{1}{k!}(tG)^k$

Proof $P_t' = G \sum_{k=1}^{+\infty} \frac{1}{(k-1)!}(tG)^{k-1} = G P_t$ ⌘

Thm $g_{ii} = (G)_{ii} \leq 0$, $g_{ij} = (G_{ij}) \geq 0$ for $i \neq j$

Proof $G = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(P_\varepsilon - I)$ #

Thm $\sum_j g_{ij} = 0$

Proof $\sum_j g_{ij} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \sum_j \left((P_\varepsilon)_{ij} - I_{ij}\right) = 0$

DEEP: Thm Chain stays $\exp(-g_{ii})$ distributed time at state i after which it moves to state j with probability $g_{ij}/(-g_{ii})$ for $j \neq i$.

stationary distribution $\pi P_t = \pi$ as before.

Thm $\pi G = 0$

Proof $\pi P_t = \pi \sum_{k=0}^{+\infty} \frac{1}{k!}(tG)^k = \pi \Leftrightarrow \pi G = 0$ #

Def Chain irreducible if $p_{ij}(t) > 0$ for some $t > 0$ for all i, j.

It turns out $p_{ij}(t) > 0$ some $t > 0 \Leftrightarrow p_{ij}(t) > 0, \forall t > 0$.

DEEP: Thm If chain is irreducible either π exists and is uniquely determined by $p_{ij}(t) = \pi_j$ as $t \to \infty$ for all i, j or else π does not exist and $p_{ij}(t) \to 0$ as $t \to \infty$ for all i, j.

<u>Example</u> The birth and death process discussed as introduction to queues in Hsu Chapter 9 is a continuous time Markov chain with

$$\begin{cases} g_{i,i+1} = \lambda_i & \text{for } i = 0, 1, \ldots \\ g_{i,i-1} = \mu_i & \text{for } i = 1, 2, \ldots \\ g_{ii} = -(\lambda_i + \mu_i) & \text{for } i = 1, 2, \ldots \\ g_{00} = -\lambda_0 & \\ g_{ij} = 0 & \text{for all other } i, j \end{cases}$$

giving

$$\pi_i = \pi_0 \frac{\lambda_0 \cdot \ldots \cdot \lambda_{i-1}}{\mu_1 \cdot \ldots \cdot \mu_i}$$