6.8.1 The time between arrivals is $\min(\exp(\lambda), \exp(\mu))$ $= \exp(\lambda+\mu)$ and after each arrival that process starts all over again. because of lack of memory property of exponential distribution.

6.8.2 As $\Psi_{\exp(\lambda)}(\omega) = E(e^{j\omega \exp(\lambda)}) = \int_0^{+\infty} e^{j\omega x} \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda - j\omega}$ the time T between arrivals of greeninsescts satisfies $\Psi_T(\omega) = E(e^{j\omega T}) = \sum_{k=1}^{+\infty} p(1-p)^{k-1} \left(\frac{\lambda}{\lambda-j\omega}\right)^k =$ $= \frac{\lambda p}{\lambda - j\omega}\left(1 - \frac{(1-p)\lambda}{\lambda - j\omega}\right)^{-1} = \lambda p / (\lambda - j\omega - (1-p)\lambda) = \frac{\lambda p}{\lambda p - j\omega}$.

6.8.5 In general $P_t' = P_t G$ and writing $p_n(t) = P(X(t)=n) = \mu_n^{(t)}$ using that $\mu^{(0)} = (1\ 0\ 0 \dots)$ we get $p_n'(t) = (\mu^{(0)} P_t)_n'$ $= (\mu^{(0)} P_t')_n = (\mu^{(0)} P_t G)_n = (P_t G)_{0n} = \sum_{k=0}^{+\infty} (P_t)_{0k} G_{kn}$ $= (P_t)_{0n} G_{nn} + (P_t)_{0(n-1)} G_{(n-1)n} = -p_n(t)(n\lambda + \nu) + p_{n-1}(t)((n-1)\lambda + \nu)$

Therefore $m(t) = E(X(t)) = \sum_{n=1}^{+\infty} n p_n(t)$ satisfies $m'(t) = \sum_{n=1}^{+\infty} n p_n'(t) = \sum_{n=1}^{+\infty} -p_n(t)(n\lambda+\nu)n + p_{n-1}(t)((n-1)\lambda+\nu)n$ $= \sum_{n=1}^{+\infty}((n-1)^2 p_{n-1}(t) - n^2 p_n(t))\lambda - \sum_{n=1}^{+\infty} \nu n p_n(t) + \sum_{n=1}^{+\infty} \lambda(n-1)p_{n-1}(t)$ $+ \sum_{n=1}^{+\infty} \nu(n-1)p_{n-1}(t) + \sum_{n=1}^{+\infty} \nu p_{n-1}(t) = \lambda m(t) + \nu$ which together with $m(0)=0$ gives ... $m(t) = \nu(e^{\lambda t} - 1)/\lambda$

6.8.6 $P(X(t)=n) = P\left(\sum_{i=1}^{n} \xi_i \le t < \sum_{i=1}^{n+1} \xi_i\right) = \int_0^t f_{\sum_{i=1}^n \xi_i}(x)\, P(\xi_{n+1} > t-x)\, dx$ $= \int_0^t (f_{\xi_1} * \dots * f_{\xi_n})(x)\, P(\xi_{n+1} > t-x)\, dx = \int_0^t (f_{\xi_1} * \dots * f_{\xi_n})(x)\, e^{-\lambda_n(t-x)} dx$ $= \lambda_n^{-1} (f_{\xi_1} * \dots * f_{\xi_{n+1}})(t)$ with characteristic function $\lambda_n^{-1} \prod_{i=0}^{n} \lambda_i / (\lambda_i - j\omega) = \sum_{i=0}^{n} \frac{a_i \lambda_i}{\lambda_i - j\omega}$ so that $P(X(t)=n)$ $= \sum_{i=0}^{n} a_i \lambda_i e^{-\lambda_i t}$ where $a_0, \dots, a_n$ is by "handpåläggning".

6.9.1 If an $n \times n$-matrix M has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and $f: \mathbb{C} \to \mathbb{C}$ is whole analytic it holds that $f(M) = \sum_{k=0}^{n-1} m_k^{(f)} M^k$ where $\sum_{k=0}^{n-1} m_k^{(f)} \lambda_i^k = f(\lambda_i)$ for $i = 1, \ldots, n$. As G has eigenvalues 0 and $-(\lambda+\mu)$ it follows that $e^{tG} = m_0 I + m_1 G$ where $m_0 + m_1 \cdot 0 = f(0) = 1$ and $m_0 - m_1(\lambda+\mu) = e^{-(\lambda+\mu)t}$ so $m_1 = \frac{1}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t})$ giving $e^{tG} = I + \frac{1}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t}) \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix} \to \frac{1}{\lambda+\mu} \begin{pmatrix} \lambda & \mu \\ \lambda & \mu \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{pmatrix}$ as $t \to \infty$ where $(\pi_1 \; \pi_2) = \left(\frac{\lambda}{\lambda+\mu} \; \frac{\mu}{\lambda+\mu} \right)$ satisfies $\pi G = 0$.

6.9.2 By conditional independence of the past and the future both probabilities are the same $P(X(0)=1, X(t)=2, X(3t)=1) / P(X(0)=1, X(3t)=1) = \mu_1^{(0)} p_{12}(t) p_{21}(2t) / (\mu_1^{(0)} p_{11}(3t))$.

6.9.3 This is simply an $M(\lambda)/M(\mu)/1$-queueing systhem with generator $G = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 \\ 0 & \mu & -(\mu+\lambda) & \lambda & 0 \\ \vdots & & \vdots & \vdots & \ddots \end{pmatrix}$ and stationary distribution $\pi_n = \pi_0 \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_0 \cdots \mu_n} = \pi_0 \left(\frac{\lambda}{\mu}\right)^n = \left(1 - \frac{\lambda}{\mu}\right)^{-1} \left(\frac{\lambda}{\mu}\right)^n$ as long as $\lambda < \mu$.

6.9.9 If $f = f_i = \sum_{n=1}^{+\infty} f_{ii}(n)$ is the probability <1 to return to i having started there and $\lambda = -g_{ii}$ the characteristic function of the time T spent in i is $\sum_{k=1}^{+\infty} (1-f) f^{k-1} \left(\frac{\lambda}{\lambda - j\omega}\right)^k$ which by exercise 6.8.2 is $\frac{\lambda(1-f)}{\lambda(1-f) - j\omega}$ so that T is $\exp(\lambda(1-f))$.

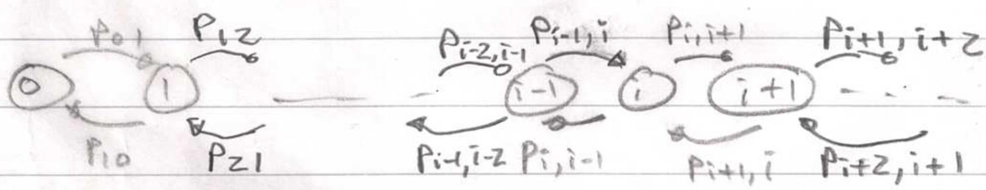
6.9.10 The probability q_i of ever visiting 0 having started at i satisfies $q_0 = 1$ and $q_i = \frac{\mu}{\lambda+\mu} q_{i-1} + \frac{\lambda}{\lambda+\mu} q_{i+1}$. As the characteristic polynomial $\frac{\lambda}{\lambda+\mu} x^2 - x + \frac{\mu}{\lambda+\mu}$ has zeros at $x = \mu/\lambda$ and $x = 1$ we get $q_i = A(\mu/\lambda)^i + B = (\mu/\lambda)^i$ as $q_0 = 1$ and $q_i \to 0$ as $i \to \infty$. Therefore the total time T_0 spent in 0 is $\exp(\lambda(1 - q_1))$ $= \exp(\lambda(1 - \frac{\mu}{\lambda})) = \exp(\lambda - \mu)$. The probability p_i of

ever returning to i having started there is
$\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}q_1=\frac{\mu}{\lambda+\mu}+\frac{\mu}{\lambda+\mu}=\frac{2\mu}{\lambda+\mu}$ so that
the total time T_i spent in i is $\exp\left((\mu+\lambda)\left(1-\frac{2\mu}{\lambda+\mu}\right)\right)$
$=\exp(\lambda-\mu)$ for $i\geq 1$. SEE TYPEWRITTEN SOLUTION!

page 4-5

6.11.1 $p_{i,i+1}=\frac{\lambda_i}{\lambda_i+\mu_i}$ and $p_{i,i-1}=\frac{\mu_i}{\lambda_i+\mu_i}$

6.11.2 The stationary distribution for the jump chain doesn't care about time spent in states which is not true for stationary distribution of original chain so therefore these need not be the same.
For the specific jump chain at hand we have
$p_{i,i+1}=\frac{\lambda}{\lambda+i\mu}$ and $p_{i,i-1}=\frac{i\mu}{\lambda+i\mu}$ with graph



giving $\pi_{i-1}p_{i-1,i}+\pi_{i+1}p_{i+1,i}=\pi_i(p_{i,i+1}+p_{i,i-1})$, $\pi_0 p_{01}=\pi_1 p_{10}$

with solution $\pi_i=\pi_0\frac{p_{01}p_{12}\cdots p_{i-1,i}}{p_{10}p_{21}\cdots p_{i,i-1}}$

$$=\pi_0\frac{1\cdot\frac{\lambda}{\lambda+\mu}\cdot\frac{\lambda}{\lambda+2\mu}\cdots\frac{\lambda}{\lambda+(i-1)\mu}}{\frac{\mu}{\lambda+\mu}\cdot\frac{2\mu}{\lambda+2\mu}\cdots\frac{i\mu}{\lambda+i\mu}}=\pi_0\left(\frac{\lambda}{\mu}\right)^i\frac{1}{\lambda}\frac{1}{i!}(\lambda+i\mu)$$

$$=\pi_0\rho^i\frac{1}{i!}(1+i/\rho)$$

6.11.4 SEE TYPE WRITEN SOLUTION! page 6

Exercise 6.9.10 in Grimmett and Stirzaker

Task. Let $X(t)$ be a birth-death process with $\lambda_n = \lambda > 0$ and $\mu_n = \mu > 0$, where $\lambda > \mu$ and $X(0) = 0$. [Or in other words, $X(t)$ is the number of customers in an M/M/1 queueing system starting up empty and with traffic intensity $\rho > 1$.] Show that the total time T_i spent in state i is $\exp(\lambda - \mu)$-distributed.

Solution. Writing q_i for the probability of ever visiting 0 having started at i we have

$$q_0 = 1 \qquad \text{and} \qquad q_i = \frac{\mu}{\lambda+\mu}\, q_{i-1} + \frac{\lambda}{\lambda+\mu}\, q_{i+1} \quad \text{for } i \geq 1.$$

The zeros of the characteristic polynomial for this difference equation are

$$p(x) = \frac{\lambda}{\lambda+\mu}\, x^2 - x + \frac{\mu}{\lambda+\mu} = 0 \;\Leftrightarrow\; x = \mu/\lambda \;\text{ or }\; x = 1,$$

```
In[3]:= Solve[lambda*x^2/(lambda+mu)-x+mu/(lambda+mu) == 0, {x}]
Out[3]= {{x -> mu/lambda}, {x -> 1}}
```

so that $q_i = A\,(\mu/\lambda)^i + B\,1^i$ for some constants $A, B \in \mathbb{R}$. As we must have $q_i \to 0$ as $i \to \infty$ we have $B = 0$ after which $q_0 = 1$ gives $A = 1$, so that $q_i = (\mu/\lambda)^i$ for $i \geq 0$

To find T_0 we note that this time is the sum of the $\exp(\lambda)$-distributed time it takes to leave 0 plus another independent $\exp(\lambda)$-distributed time added for each revisit of 0, where the number N of such revisits has PMF $\mathbf{P}\{N = n\} = (\mu/\lambda)^n(1-\mu/\lambda)$ for $n \geq 0$. As the CHF of an $\exp(\lambda)$-distributed random variable is $\mathbf{E}\{e^{j\omega \exp(\lambda)}\} = \lambda/(\lambda - j\omega)$

```
In[4]:= Integrate[Exp[I*omega*x]*lambda*Exp[-lambda*x],
                  {x,0,Infinity}, Assumptions->lambda>0&&omega∈Reals]
Out[4]= lambda/(lambda - I omega)
```

it follows that (making use of the basic fact that the CHF of a sum of independent random variables is the product of the CHF's of the individual random variables)

$$\mathbf{E}\{e^{j\omega T_0}\} = \frac{\lambda}{\lambda - j\omega} \times \sum_{n=0}^{\infty} \Big(\frac{\lambda}{\lambda - j\omega}\Big)^n (\mu/\lambda)^n (1-\mu/\lambda) = \frac{\lambda - \mu}{\lambda - \mu - j\omega},$$

```
In[5]:= lambda/(lambda-I*omega)*Sum[(lambda/(lambda-I*omega))^n*
                  (mu/lambda)^n*(1-mu/lambda), {n,0,Infinity}]
Out[5]= (lambda - mu)/(lambda - mu - I omega)
```

To find T_i we note that (by considering what the first state after having left i is – $i-1$ or $i+1$) the probability of ever returning to i having started there is

$$\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \cdot q_1 = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \cdot \frac{\mu}{\lambda} = \frac{2\,\mu}{\lambda+\mu}.$$

As the time spent at each visits of i is $\exp(\lambda+\mu)$-distributed it follows as above that

$$\mathbf{E}\{e^{j\omega T_i}\} = \frac{\lambda+\mu}{\lambda+\mu-j\omega} \times \sum_{n=0}^{\infty} \Big(\frac{\lambda+\mu}{\lambda+\mu-j\omega}\Big)^n \Big(\frac{2\,\mu}{\lambda+\mu}\Big)^n \Big(1 - \frac{2\,\mu}{\lambda+\mu}\Big) = \frac{\lambda-\mu}{\lambda-\mu-j\omega},$$

```
In[6]:= (lambda+mu)/(lambda+mu-I*omega)*
                    Sum[(2*mu/(lambda+mu-I*omega))^n*
                            (1-2*mu/(lambda+mu)), {n,0,Infinity}]

Out[6]= (lambda - mu)/(lambda - mu - I omega)
```

Exercise 6.11.4 in Grimmett and Stirzaker

Task. Let $X(t)$ be a birth-death process with $\lambda_n = n\lambda$ and $\mu_n = n\mu$, where $0 < \lambda < \mu$ and $X(0) = 1$. Show that the distribution of $X(t)$ conditional on the event that $\{X(t) > 0\}$ converges as $t \to \infty$ to a geometric distribution.

Solution. By Theorem 6.11.10 in G&S $X(t)$ has probability generating function

$$G(s,t) = \sum_{n=0}^{\infty} s^n \, \mathbf{P}\{X(t) = n\} = \frac{\mu\,(1-s) - (\mu - \lambda s)\,\mathrm{e}^{-t(\lambda-\mu)}}{\lambda\,(1-s) - (\mu - \lambda s)\,\mathrm{e}^{-t(\lambda-\mu)}}.$$

The probability generating function of $X(t)$ conditional on that $\{X(t) > 0\}$ is therefore

$$\begin{aligned}
\sum_{n=1}^{\infty} s^n \, \mathbf{P}\{X(t) = n \,|\, X(t) > 0\} &= \sum_{n=1}^{\infty} s^n \, \frac{\mathbf{P}\{X(t) = n\}}{\mathbf{P}\{X(t) > 0\}} \\
&= \frac{G(s,t) - \mathbf{P}\{X(t) = 0\}}{\mathbf{P}\{X(t) > 0\}} \\
&= \frac{G(s,t) - G(0,t)}{1 - G(0,t)} \\
&= \frac{(\mu - \lambda)\, s\, \mathrm{e}^{-t(\lambda-\mu)}}{(\mu - \lambda s)\,\mathrm{e}^{-t(\lambda-\mu)} - \lambda(1-s)} \\
&\to \frac{(\mu - \lambda)\, s}{\mu - \lambda s} \quad \text{as } t \to \infty \\
&= \frac{(1-\rho)\, s}{1 - \rho s} \quad \text{where } \rho = \lambda/\mu \\
&= (1-\rho)\, s \sum_{n=0}^{\infty} (\rho s)^n \\
&= \sum_{n=1}^{\infty} s^n \, \rho^{n-1} (1-\rho)
\end{aligned}$$

```
In[7]:= Clear[G]; G[s_,x_] := (mu*(1-s)-(mu-lambda*s)*x)/
                            (lambda*(1-s)-(mu-lambda*s)*x)

In[8]:= Simplify[(G[s,x]-G[0,x])/(1-G[0,x])]

Out[8]= ((mu-lambda)*s*x) / ((mu-lambda*s)*x-lambda*(1-s))
```

so that $\mathbf{P}\{X(t) = n \,|\, X(t) > 0\} = \rho^{n-1}(1-\rho)$ for $n \geq 1$ indeed is geometrically distributed.