Sections 5.1-5.4 in Hsu

Def A random process (also called stochastic process) is a family $X(t) = X(t, \zeta)$, $\zeta \in S$, of random variables indexed by time $t \in T$

The time parameter set T is either discrete $T = \mathbb{N}, \mathbb{Z}, \{0, \dots, n\}$ etc or continuous $T = \mathbb{R}, \mathbb{R}^+, [a, b]$ etc.

As the CDF $F_X(x)$ is used to study r.v.'s X one might believe the CDF $F_{X(t)}(x)$ of a random process $X(t)$ at each time t says a lot about the process.

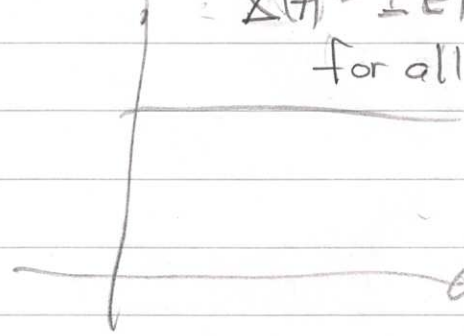
This is not true:

Ex 1

all $X(t)$ independent $N(0,1)$-distributed

Ex 2

$X(t) = Y \in N(0,1)$ for all t



Both the above processes have CDF $F_{X(t)}(x) = \Phi(x)$ for each $t \in T$ but are totally different in behaviour.

Def The mean function $\mu_X(t) = E(X(t))$

The autocorrelation function $R_X(t_1,t_2) = E(X(t_1)X(t_2))$

The autocovariance function $K_X(t_1,t_2) = Cov(X(t_1),X(t_2))$

Crosscorrelation function $R_{X,Y}(t_1,t_2) = E(X(t_1)Y(t_2))$

Crosscovariance function $K_{X,Y}(t_1,t_2) = Cov(X(t_1),Y(t_2))$

Example For $X(t) = U\cos(\omega t) + V\sin(\omega t)$ where U and V are zero mean uncorrelated with variance σ^2 we have $\mu_X(t) = 0$ and

$$R_X(s,t) = C_{X,X}(s,t) = E(U^2)\cos(\omega s)\cos(\omega t) + E(V^2)\sin(\omega s)\sin(\omega t) = \sigma^2\cos(\omega(s-t)) \quad \#$$

cosine-process

Example For $X(t) = a(\sin\omega t + \Theta)$ with Θ uniformly distributed over $[0,2\pi]$ we have $\mu_X(t) = 0$ and

$$R_X(s,t) = C_{X,X}(s,t) = a^2 E(\sin(\omega s + \Theta)\sin(\omega t + \Theta))$$

$$= a^2 E\left(\tfrac{1}{2}\cos(\omega s - \omega t) + \tfrac{1}{2}\cos(\omega s + \omega t + 2\Theta)\right)$$

$$= \tfrac{1}{2}a^2\cos(\omega s - \omega t) \quad \#$$

Def $X(t)$ is (strictly) stationary if $F_{X(t_1),\dots,X(t_n)}(x_1,\dots,x_n) = F_{X(t_1+h),\dots,X(t_n+h)}(x_1,\dots,x_n)$

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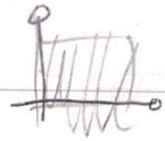
Example The processes  and  are both stationary.

Def $X(t)$ is wide sense stationary WSS (weak) if

$$\begin{cases} \mu_X(t) = \mu_X \\ R_{XX}(t, t+\tau) \end{cases}$$ does not depend on t.

Thm stationary $\Rightarrow$ WSS

Notation For $X(t)$ WSS we write $R_X(t, t+\tau) = R_X(\tau)$.

Example Both trigonometric processes we saw earlier are WSS, as are and

Properties of autocorrelation for WSS $X(t)$

* $|R_X(\tau)| \leq R_X(0)$ and $\left(E(X(t)X(t+\tau))\right)^2 \leq \left(E(X(t)^2)\right)\left(E(X(t+\tau)^2)\right)$

since $E\left(\left(\frac{X(t)}{\sqrt{R_{XX}(0)}} \pm \frac{X(t+\tau)}{\sqrt{R_{XX}(0)}}\right)^2\right) \geq 0$

* $R_X(\tau) = R_X(-\tau)$ * $R_X(0) = E(X(t)^2)$

Def A random process $X(t)$ is Gaussian if $\sum_{i=1}^{n} a_i X(t_i)$ is normal for all a_i and t_i

Note $X(t)$ Gaussian $\Rightarrow$ each $X(t)$-value normal but it is not the other way around!

Ex $X(t) = U\cos(\omega t) + V\sin(\omega t)$ with $U, V \in N(0, \sigma^2)$ independent is Gaussian.

Fact A Gaussian process is (probabilistically) determined entirely by its mean function and autocorrelation function.

Proof* $\varphi_{X(t_1),\dots,X(t_n)}(\omega_1,\dots,\omega_n) = E\left(e^{j\sum_{i=1}^{n}\underbrace{\omega_i X(t_i)}_{N(\mu,\sigma^2)}}\right)$

where $\mu = E\left(\sum_{i=1}^{n}\omega_i X(t_i)\right) = \sum_{i=1}^{n}\omega_i \mu_X(t_i)$ and

$\sigma^2 = Var\left(\sum \omega_i X(t_i)\right) = \sum_i \sum_j \omega_i \omega_j K_X(t_i,t_j)$. #

determined by R_{XX} and μ_X

Fact A Gaussian process is stationary if WSS.

Proof If WSS it has same μ_X and R_{XX} as stationary and therefor is stationary as determined by those. #

Fact Two Gaussian process values $X(s)$ and $X(t)$ are independent if uncorrelated.

Proof If uncorrelated they have same μ_X and $R_{XX}(s,t)$ as if independent and thus are independent since determined by those. #

Example Find $Pr(X(1)+2X(2) \geq 3)$ for cosine-process with U and V Gaussian/normal.

Solution $X(1)+2X(2)$ is $N(0,\Sigma^2)$ with

$$\Sigma^2 = Var(X(1)+2X(2)) = Var(X(1)) + 4\,Cov(X(1),X(2)) + 4\,Var(X(2))$$
$$= \sigma^2 + 4\sigma^2\cos(1) + 4\sigma^2$$

so

$$Pr(X(1)+2X(2)\geq 3) = 1 - Pr(N(0,\Sigma^2)\leq 3)$$
$$= 1 - \Phi\left(\frac{3-0}{\sqrt{5\sigma^2+4\sigma^2\cos(1)}}\right).$$

We do not discuss subsection 5.4.C about <u>independent processes</u> as uninteresting. (This was one of our first examples of a random process.)

We discuss <u>discrete valued Markov processes</u> called <u>Markov chains</u> in Section 5.5 so we jump subsection 5.4.E as well.

We do not discuss subsection 5.4.G on <u>ergodic processes</u> as better suited to cover in second more advanced random process course.

<u>Subsection 5.4.D</u>

Def A random process $\{X(t)\}_{t\geq 0}$ has <u>stationary independent increments</u> (also called <u>Lèvy process</u>) if

* $X(0) = 0$
* the probability distribution of $X(t+s) - X(s)$ depends on t only but not on s for $0 \leq s \leq t+s$
* $X(t) - X(s)$ is independent of $\{X(r)\}_{r\in[0,s]}$ of $0 \leq s \leq t$

<u>Ex</u> The two most important stationary independent increment processes the Poisson process and the Wiener process we meet in Sections 5.6-5.7.