

Section 5.5 in Hsu

①

DEF

A discrete time discrete valued random process $\{X_n\}_{n=0}^{+\infty}$ is called a Markov chain (and has the Markov property) if for all x_0, \dots, x_{n+1} and n

$$P(X_{n+1} = x_{n+1} | X_n = x_n, \dots, X_0 = x_0) = P(X_{n+1} = x_{n+1} | X_n = x_n)$$

Markov chains are the discrete time special case of more general Markov processes. We study these processes only for discrete time and discrete values.

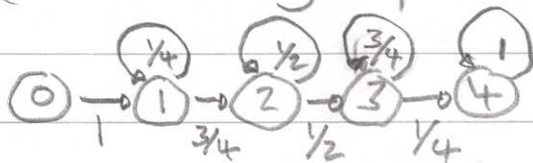
We assume without loss that our Markov chains are integer \mathbb{Z} -valued as they can otherwise be coded to be so.

DEF

The transition probabilities $p_{ij} = P(X_{n+1} = j | X_n = i)$ are elements of the transition matrix $P = (p_{ij})$.

We always assume so called time homogeneity that p_{ij} does not depend on n .

Example (Kid collecting super hero figures at fast food restaurant)

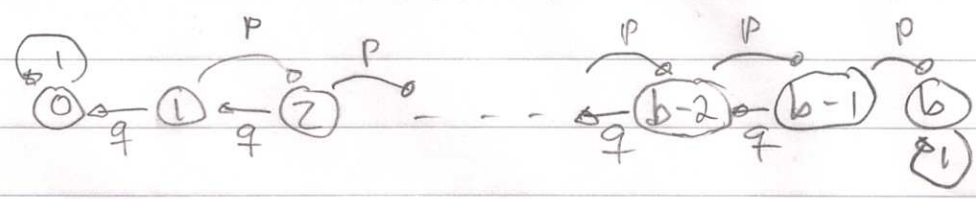


$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 3/4 & 1/4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

X_n = number of different super hero figures out of four possible collected after n restaurant visits

Example (Random walk) $X_0 = 0$ and $X_n = \sum_{i=1}^n Y_i$ where $\{Y_i\}_{i=1}^{+\infty}$ are independent identically distributed with $P(Y_i = 1) = p$ and $P(Y_i = -1) = 1 - p = q$.
 The transition matrix is $p_{i,i+1} = p$, $p_{i,i-1} = q$ and $p_{ij} = 0$ for $j \notin \{i-1, i+1\}$.

Example (Gamblers ruin problem) Gambler initially has $X_0 = d$ dollars while house has $b - d$ dollars



DEF The distribution at time n row matrix $p^{(n)}$ has elements $p^{(n)}_j = P(X_n = j)$.

DEF The n-step transition probabilities $p^{(n)}_{ij} = P(X_{m+n} = j | X_m = i)$ are elements of the n-step transition matrix $P^{(n)} = (p^{(n)}_{ij})$.

THM (Chapman-Kolmogorov) $P^{(n)} = P^n$

Proof

$$(P^{(n+1)})_{ij} = P(X_{m+n+1} = j | X_m = i, \dots) = \frac{P(X_{m+n+1} = j, X_m = i, \dots)}{P(X_m = i, \dots)}$$

$$= \sum_{\text{all } k} \frac{P(X_{m+n+1} = j, X_{m+1} = k, X_m = i, \dots)}{P(X_{m+1} = k, X_m = i, \dots)} \cdot \frac{P(X_{m+1} = k, X_m = i, \dots)}{P(X_m = i, \dots)}$$

$$= \sum_{\text{all } k} (P^{(n)})_{kj} P_{ik} \Rightarrow P^{(n+1)} = P P^{(n)} = P P P^{(n-1)} = \dots = P^n \quad \#$$

The Markov property is equivalent with the requirement that for all $i_0 < i_1 < \dots < i_{n+1}$, x_0, \dots, x_{n+1} and n
 $P(X_{i_{n+1}} = x_{n+1} | X_{i_n} = x_n, \dots, X_{i_0} = x_0) = P(X_{i_{n+1}} = x_{n+1} | X_{i_n} = x_n)$.

Let us just check this for $i_0=0, \dots, i_n=n, i_{n+1}=n+2$:

$$\begin{aligned}
 P(X_{n+2}=y | X_n=x_n, \dots, X_0=x_0) &= \frac{P(X_{n+2}=y, X_n=x_n, \dots, X_0=x_0)}{P(X_n=x_n, \dots, X_0=x_0)} \\
 &= \sum_{\text{all } x} \frac{P(X_{n+2}=y, X_{n+1}=x, X_n=x_n, \dots, X_0=x_0)}{P(X_{n+1}=x, X_n=x_n, \dots, X_0=x_0)} \frac{P(X_{n+1}=x, X_n=x_n, \dots, X_0=x_0)}{P(X_n=x_n, \dots, X_0=x_0)} \\
 &= \sum_{\text{all } x} P(X_{n+2}=y | X_{n+1}=x, X_n=x_n, \dots, X_0=x_0) P(X_{n+1}=x | X_n=x_n, \dots, X_0=x_0) \\
 &= \sum_{\text{all } x} P_{xy} P_{x_n x} = (P^{(2)})_{x_n y} \text{ (by Chapman-Kolmogorov)}.
 \end{aligned}$$

[T+M]

$$p^{(m+n)} = p^{(m)} p^{(n)} = p^{(m)} P^n$$

Proof

$$\begin{aligned}
 p^{(m+n)}_j &= P(X_{m+n}=j) = \sum_{\text{all } k} P(X_{m+n}=j, X_m=k) \\
 &= \sum_{\text{all } k} p^{(m)}_k (P^{(n)})_{kj} = (p^{(m)} P^{(n)})_j \neq
 \end{aligned}$$

[DEF]

A row matrix \hat{p} is called a stationary distribution for a Markov chain if $\hat{p} P = \hat{p}$, $\hat{p}_i \geq 0$ and $\sum_{\text{all } i} \hat{p}_i = 1$.

Fact

If $p^{(m)} = \hat{p}$ then $p^{(m+n)} = \hat{p}$ for $n \geq 1$.

Proof

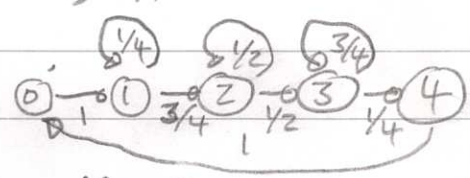
$$p^{(m+n)} = p^{(m)} P^n = p^{(m)} P^n = \hat{p} P P^{n-1} = \hat{p} P^{n-1} = \dots = \hat{p} \neq$$

Example

For super heroes $\hat{p} = (0 \ 0 \ 0 \ 0 \ 1)$ #

Example

(Modified super heroes)



$$P = \begin{pmatrix}
 0 & 1 & 0 & 0 & 0 \\
 0 & 1/4 & 3/4 & 0 & 0 \\
 0 & 0 & 1/2 & 1/2 & 0 \\
 0 & 0 & 0 & 3/4 & 1/4 \\
 1 & 0 & 0 & 0 & 0
 \end{pmatrix}$$

$$\hat{p} P = \hat{p} \Leftrightarrow \begin{cases} \hat{p}_4 = \hat{p}_0 \\ \hat{p}_0 + 1/4 \hat{p}_1 = \hat{p}_1 & \hat{p}_1 = 4/3 \hat{p}_0 \\ 3/4 \hat{p}_1 + 1/2 \hat{p}_2 = \hat{p}_2 & \hat{p}_2 = 2 \hat{p}_0 \\ 1/2 \hat{p}_2 + 3/4 \hat{p}_3 = \hat{p}_3 & \hat{p}_3 = 4 \hat{p}_0 \\ 1/4 \hat{p}_3 = \hat{p}_4 & \hat{p}_4 = \hat{p}_0 \end{cases}$$

$$\sum_{\text{all } i} \hat{p}_i = \hat{p}_0 (1 + 4/3 + 2 + 4 + 1) = 28/3 \Rightarrow \hat{p} = \left(\frac{3}{28} \quad \frac{4}{28} \quad \frac{6}{28} \quad \frac{12}{28} \quad \frac{3}{28} \right)$$

DEF The mean time to return to a state (= possible value) i of the Markov chain is defined

$$\mu_i = E(\min\{n \geq 1 : X_n = i\} | X_0 = i)$$

THM For an irreducible aperiodic (see below) Markov chain it holds that \hat{p} exists if and only if $\mu_i < \infty$ for all i and in that case $\hat{p}_i = 1/\mu_i$ for all i .

Example (Super heroes) As $\mu_0 = 1/\hat{p}_0 = 28/3$ for modified super hero we have $E(\min\{n \geq 1 : X_n = 4\} | X_0 = 0) = 25/3$ for original super hero. This can also be concluded from using that mean of waiting time distribution is $1/p$ as $1 + 1/3 + 2 + 4 = 25/3$.

DEF j is accessible from i ($i \rightarrow j$) if $p_{ij}^{(n)} > 0$ for some n
 i and j communicate ($i \leftrightarrow j$) if $i \rightarrow j$ and $j \rightarrow i$
the chain is irreducible if $i \leftrightarrow j$ for all i and j

Example Random walk and modified super hero are irreducible while original super hero is not.

DEF $d(i) = \text{GCD}(\{n \geq 1 : p_{ii}^{(n)} > 0\})$ is the period of i .
The chain is aperiodic if $d(i) = 1$ for all i

Example Random walk has $d(i) = 2$ for all i while modified super hero has $d(i) = 1$ for all i .

DEF i is transient if $P(X_n = i \text{ some } n \geq 1 | X_0 = i) < 1$
 i is recurrent if $P(X_n = i \text{ some } n \geq 1 | X_0 = i) = 1$

THM

i is recurrent if and only if $\sum_{n=1}^{+\infty} p_{ii}^{(n)} = \infty$

THM

For an irreducible chain either all states are recurrent or all states are transient. Further all states have same period and $\mu_i < \infty$ for some i if and only if $\mu_i < \infty$ for all i .

Example

For random walk $p_{ii}^{(n)} = \binom{n}{n/2} p^{n/2} (1-p)^{n/2}$ for n even while $p_{ii}^{(n)} = 0$ for n uneven. Using Stirling's formula $n! \sim \sqrt{2\pi n} n^n e^{-n}$ as $n \rightarrow \infty$

we see that

$$p_{ii}^{(2n)} = \frac{(2n)! (p(1-p))^n}{(n!)^2} \sim \frac{\sqrt{4\pi n} (2n)^{2n} e^{-2n} (p(1-p))^n}{(2\pi n) n^{2n} e^{-2n}} = \frac{(4 \cdot p(1-p))^n}{\sqrt{\pi n}}$$

so that $\sum_{n=1}^{+\infty} p_{ii}^{(2n)} = \infty$ if and only if $p = 1/2$.