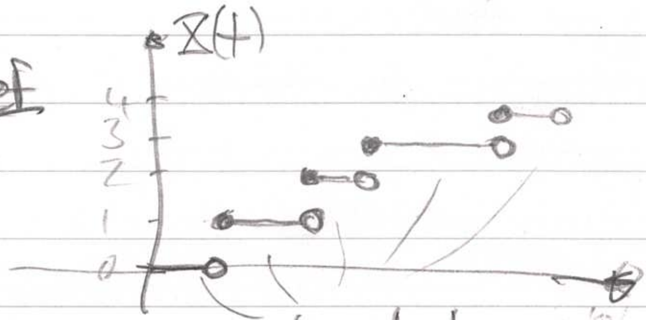
Section 5.6–5.7 in Hsu

5.6 Poisson process

Def Poisson process $X(t)$, $t \geq 0$

1. $X(0) = 0$
2. $X(t+s) - X(s) \sim Po(\lambda t)$ for $s, t \geq 0$
3. $X(t+s) - X(s)$ independent of $(X(r))_{r \in [0,s]}$

Alt def



independent exp-distributed with mean $\frac{1}{\lambda}$

Example $\mu_X(t) = \lambda t$

$$R_X(t, t+\tau) = E(X(t)X(t)) + E(X(t)(X(t+\tau) - X(t)))$$

$$= E(Po(\lambda t)^2) + E(Po(\lambda t))E(Po(\lambda\tau))$$

$$= \lambda t + (\lambda t)^2 + \lambda t \lambda \tau$$

$\boxed{\text{Follows also from Exercises 5.22–5.23}}$

Example $$P(X(1)=1 \mid X(2)=2) = \frac{P(X(1)=1, X(2)=2)}{P(X(2)=2)}$$

$$= \frac{P(X(1)=1)\,P(X(2)-X(1)=1)}{P(X(2)=2)} = \frac{\frac{\lambda^1}{1!}e^{-\lambda}\frac{\lambda^1}{1!}e^{-\lambda}}{\frac{(2\lambda)^2}{2!}e^{-2\lambda}} = \frac{1}{2}$$

5.7 Wiener processes

Def A random process $(X(t))_{t\geq 0}$ is a <u>Wiener process</u> (also called <u>Brownian motion</u>) if it is a stationary independent increment process with $X(t)-X(s)$ Normal$(0,\sigma^2(t-s))$ distributed for $0\leq s\leq t$.

Thm A Wiener process is a zero-mean Gaussian process with

$$R_X(s,t) = K_X(s,t) = \sigma^2 \min(s,t).$$

<u>Proof</u> $X(t) = (X(t)-X(0)) + X(0) = N(0,\sigma^2 t + 0)$ is clearly zero-mean with variance $\sigma^2 t$. Hence it follows from Exercise 5.23 what R_X/K_X is.

To show that $X(t)$ is Gaussian note

$$\begin{aligned}\sum_{i=1}^{n} a_i X(t_i) = {} & a_n(X(t_n)-X(t_{n-1})) + \\ & + (a_n+a_{n-1})(X(t_{n-1})-X(t_{n-2})) \\ & + (a_n+a_{n-1}+a_{n-2})(X(t_{n-2})-X(t_{n-3})) \\ & + \cdots \\ & + (a_n+\cdots+a_2)(X(t_2)-X(t_1)) \\ & + (a_n+\cdots+a_1)X(t_1)\end{aligned}$$

for $0\leq t_1\leq\ldots\leq t_n$ which is a sum of independent normal random variables by definition of $X(t)$ and thus again a normal random variable. #