Hsu Section 5.8: Martingales

More on conditional expectation

* In undergraduate probability one defines

$$E(Y|X_0=x_0,\dots,X_n=x_n) = \begin{cases} \int_{-\infty}^{+\infty} y f_{Y|X_0,\dots,X_n}(y|x_0,\dots,x_n)\,dy & \text{for continuous r.v.'s} \\ \sum_{k=-\infty}^{+\infty} k f_{Y|X_0,\dots,X_n}(k|x_0,\dots,x_n) & \text{for discrete r.v.'s} \end{cases}$$

where

$$f_{Y|X_0,\dots,X_n}(y|x_0,\dots,x_n) = \frac{f_{Y,X_0,\dots,X_n}(y,x_0,\dots,x_n)}{f_{X_0,\dots,X_n}(x_0,\dots,x_n)}$$

and then we have

$$E(Y|X_0=x_0,\dots,X_n=x_n) = g(x_0,\dots,x_n) \leftarrow \text{a function } g:\mathbb{R}^{n+1}\to\mathbb{R} \text{ of } \bar{x}$$

* In more advanced probability one defines

$$E(Y|X_0,\dots,X_n) = g(X_0,\dots,X_n) \leftarrow \text{a random variable where } g \text{ is as above}$$

This can be viewed as the information that $(X_0,\dots,X_n)=(x_0,\dots,x_n)$ is not inserted but one keeps the random values of $X_0,\dots,X_n$ themselves and that is the information conditioned on.

* We use the short hand notation $\underline{F_n}$ for the information $X_0,\dots,X_n$ and we also use $\sigma(X_0,\dots,X_n)$ to denote information $F_n=\sigma(X_0,\dots,X_n)$

* $E(Y \mid X_0, \dots, X_n) = E(Y \mid \sigma(X_0, \dots, X_n)) = E(Y \mid F_n)$
$$= g(X_0, \dots, X_n)$$
where
$$g(x_0, \dots, x_n) = E(Y \mid X_0 = x_0, \dots, X_n = x_n)$$

* A r.v. Y is called <u>F_n-measurable</u> if it can be written as a function of $X_0, \dots, X_n$.

<u>Thm</u> ① $E(aY_1 + bY_2 \mid F_n) = aE(Y_1 \mid F_n) + bE(Y_2 \mid F_n)$
② $E(Y \mid F_n) \geq 0$ for $Y \geq 0$
③ $E(Y \mid F_n) = Y$ for Y F_n-measurable
④ $E(ZY \mid F_n) = Z\,E(Y \mid F_n)$ for Z F_n-measurable
⑤ $E(Y \mid F_n) = E(Y)$ for Y independent of F_n
⑥ $E(E(Y \mid F_n) \mid F_m) = E(Y \mid F_m)$ for $m \leq n$
⑦ $E(E(Y \mid F_n)) = E(Y)$
⑧ $E(g(Y) \mid F_n) \geq g(E(Y \mid F_n))$ for g convex

<u>Proof</u> ①-② and ⑧ proved as for ordinary expetation.
③-⑤ by inspection of definition.
⑥ Omitted.
⑦ $\displaystyle\int_{x_0=-\infty}^{x_0=+\infty} \cdots \int_{x_n=-\infty}^{x_n=+\infty} \int_{y=-\infty}^{y=+\infty} y \frac{f_{Y, X_0, \dots, X_n}(y, x_0, \dots, x_n)}{f_{X_0, \dots, X_n}(x_0, \dots, x_n)} f_{X_0, \dots, X_n}(x_0, \dots, x_n)\, dy\, d\bar{x} = E(Y)$ #

* Until further notice $\{X_n\}_{n=0}^{+\infty}$ is a discrete time random process, $F_n = \sigma(X_0, \dots, X_n)$ and we call $\{F_n\}_{n=0}^{+\infty}$ a <u>filtration</u>. (Not important really.)

$\boxed{\text{Def}}$ $\{X_n\}_{n=0}^{+\infty}$ is a <u>martingale</u> wrt. $\{F_n\}_{n=0}^{+\infty}$ if
① $E(|X_n|) < \infty$ for $n \geq 0$
② $E(X_{n+1} \mid F_n) = X_n$ for $n \geq 0$

Def $\{X_n\}_{n=0}^{+\infty}$ is a <u>submartingale</u> (<u>supermartingale</u>) wrt. $\{F_n\}_{n=0}^{+\infty}$ if

$E\{|X_n|\} < \infty$ for $n \geq 0$

$E(X_{n+1} | F_n) \geq (\leq) X_n$ for $n \geq 0$

Thm For $\{X_n\}_{n=0}^{+\infty}$ a martingale

① $E(X_n) = E(X_0)$ for $n \geq 0$

② $E(X_{m+n} | F_n) = X_n$ for $m \geq 1$

③ $\{g(X_n)\}_{n=0}^{+\infty}$ is a submartingale for g convex

Proof ① $E(X_{n-1}) = E(E(X_n | F_{n-1})) = E(X_n)$ for $n \geq 1$

② $E(X_{m+n} | F_n) = E(E(X_{m+n} | F_{m+n-1}) | F_n) = E(X_{m+n-1} | F_n)$

$= \dots = E(X_{n+1} | F_n) = X_n$ for $m \geq 2$

③ $E(g(X_{n+1}) | F_n) \geq g(E(X_{n+1} | F_n)) = g(X_n)$ #

Def An $\mathbb{N} \cup \{+\infty\}$-valued r.v. T is a <u>stopping time</u> if $\{T=n\}$ is F_n-measurable (i.e., if one can say whether $\{T=n\}$ or not using information $\sigma(X_0, \dots, X_n)$)

Thm (Optional Stopping Theorem) not optimal For $\{X_n\}_{n=0}^{+\infty}$ a martingale and T a stopping time such that

① $E(T) < \infty$

② $E(|X_T|) < \infty$

③ $\lim_{n \to \infty} E(|X_n| 1_{\{T > n\}}) = 0$

we have

$$E(X_T) = E(X_0)$$

Proof Difficult!

Example (Computational excerise 3). Let

$$X_0 = 100 \text{ and } X_n = X_0 + \sum_{i=1}^{n} Y_i \text{ for } n \geq 1$$

where $Y_1, \dots, Y_n$ are iid random variables with

$$P(Y_i = 4) = 1/5 \quad \text{and} \quad P(Y_i = -1) = 4/5$$

Then we have

$$E(X_{n+1} | F_n) = E(Y_{n+1} + X_n | F_n) = E(Y_{n+1} | F_n) + E(X_n | F_n)$$
$$= E(Y_{n+1}) + X_n = 4 \cdot \frac{1}{5} + (-1) \cdot \frac{4}{5} + X_n = X_n$$

so X_n is martingale. Letting

$$T = \min\{n \geq 1 : X_n = 0 \text{ or } X_n \geq 200\}$$

it is clear that T is a stoppingtime and that conditions ①–③ of optional stopping theorem holds. (See also extra material on exercise web-page.) Hece

$$100 = E(X_0) = E(X_T) = E(X_T 1_{\{X_T = 0\}} + X_T 1_{\{X_T \geq 200\}})$$
$$= 0 + E(X_T 1_{\{X_T \geq 200\}}) \begin{cases} \leq 203 \cdot P\{X_T \geq 100\} \\ \geq 200 \cdot P\{X_T \geq 100\} \end{cases}$$

so that $P\{X_T \geq 200\} \in \left[\frac{100}{203}, \frac{100}{200}\right]$.

Continuous time martingales

Def A continuous time random process $\{X(t)\}_{t\geq 0}$ is a martingale wrt. the filtration $\{F_t\}_{t\geq 0}$ where F_t is the information $\sigma(\{X(s)\}_{s\in[0,t]})$ of the history of the process up to time t if

① $E(|X(t)|) < \infty$ for $t \geq 0$

② $E(X(t)|F_s) = X(s)$ for $0 \leq s \leq t$

Thm $E(X(t)) = E(X(0))$ for $t \geq 0$

$g(X(t))$ is a submartingale for g convex

Proof Same as before. #