Hsu Sections 6.1–6.3B

Hsu 6.2: Continuity, differentiation, integration

To discuss above topics for random processes remember that they are defined using limits so we must discuss limits of random variables.

Def $X_n \to X$ in mean-square also denoted $\text{l.i.m.}\, X_n = X$ if $E(X^2) < \infty$ and $\lim_{n\to\infty} E((X_n - X)^2) = 0$.

Thm $\text{l.i.m.}\, X_n = X$ and $\text{l.i.m.}\, Y_n = Y$ implies $E(X_n) \to E(X)$, $E(X_n^2) \to E(X^2)$ and $E(X_n Y_n) \to E(XY)$

Part proof $|E(X_n) - E(X)| \overset{\triangle}{\leq} E(|X_n - X|) \overset{\text{Jensen}}{\leq} \sqrt{E((X_n - X)^2)}$

$|E(X_n^2) - E(X^2)| = |E(X_n(X_n - X)) + E(X(X_n - X))|$

$\overset{\triangle}{\leq} E(|X_n||X_n - X|) + E(|X||X_n - X|) \overset{CS}{\leq} \sqrt{E(X_n^2)E((X_n - X)^2)}$

$+ \sqrt{E(X^2)E((X_n - X)^2)}$ #

Def A continuous time process $X(t)$ is continuous at $t = t_0$ if $\text{l.i.m.}_{t\to t_0} X(t) = X(t_0)$

Thm $X(t)$ is continuous at $t = t_0$ if (and only if) $R_X(s,t)$ is continuous at $(s,t) = (t_0, t_0)$ and in that case $\mu_X(t)$ is continuous at $t = t_0$.

Proof $E((X(t) - X(t_0))^2) = R_X(t,t) - 2R_X(t,t_0) + R_X(t_0,t_0)$

and $|\mu_X(t) - \mu_X(t_0)| \leq E(|X(t_0) - X(t)|) \leq \sqrt{E((X(t) - X(t_0))^2)}$

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<u>Example</u> Stationary independent increment processes are continuous as $R_X(s,t) = E(X(s)X(t)) = K_X(s,t) + \mu_X(s)\mu_X(t) = Var(X(1))\min(s,t) + (E(X(1)))^2 st$ by solved exercises in Chapter 5, which is continuous. Note that this means that Poisson process is continuous.

$\boxed{\text{Def}}$ A continuous time process <u>$X(t)$ is differentiable at $t=t_0$</u> with derivative $X'(t_0)$ if $\text{l.i.m.}_{h\to 0} \frac{1}{h}(X(t_0+h) - X(t_0)) = X'(t_0)$

<u>Thm</u> $X(t)$ is differentiable if $\frac{\partial^2}{\partial s \partial t} R_X(s,t)$ exists and in that case $R_{X'}(s,t)$ is equal to that mixed dervivative.

<u>Proof</u> See the exercises.

<u>Example</u> Stationary independent increment processes are not differentiable as $\min(s,t)$ is not differentiable as $\frac{\partial}{\partial t}\min(s,t) = 0$ for $s<t$ and 1 for $s>t$ is not continuous.

$\boxed{\text{Def}}$ $\int_a^b X(t)\,dt = \text{l.i.m.}\left\{\sum_{i=1}^n X(\xi_i)(t_i - t_{i-1}) : \begin{array}{l} a=t_0<t_1<\ldots<t_n=b \\ \max_{1\le i\le n} t_i - t_{i-1} \to 0, \xi_i \in [t_{i-1}, t_i] \end{array}\right\}$

<u>Thm</u> $E\left(\int_a^b X(s)\,ds \int_c^d Y(t)\,dt\right) = \int_a^b\int_c^d R_{XY}(s,t)\,dt\,ds$

<u>Proof</u> As the coresponding statement for sums.

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Hsu Section 6.3 A–B

The properties of the autocorrelation function $R_X(\tau)$ for a WSS process $X(t)$ in Section 6.3 A we discussed in the lecture on Hsu Sections 5.1–5.4.

Now consider two WSS processes $X(t)$ and $Y(t)$ that are jointly WSS, which is to say that $R_{XY}(t, t+\tau) = E(X(t)Y(t+\tau)) = R_{XY}(\tau)$ depends on τ only.

Thm ① $R_{XY}(-\tau) = R_{YX}(\tau)$

② $|R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)}$

③ $|R_{XY}(\tau)| \leq \frac{1}{2}(R_X(0) + R_Y(0))$

Proof ① By inspection.

②
$$0 \leq E\left(\left(\frac{X(t)}{\sqrt{R_X(0)}} \pm \frac{Y(t+\tau)}{\sqrt{R_Y(0)}}\right)^2\right) = 1 \pm 2\frac{R_{XY}(\tau)}{\sqrt{R_X(0)R_Y(0)}} + 1$$

③
$$0 \leq E((X(t) \pm Y(t+\tau))^2) = R_X(0) \pm 2R_{XY}(\tau) + R_Y(0)$$

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