# Hsu Section 6.3C-6.5: PSD's, white noise & LTI

Consider a continuous time WSS process $\{X(t)\}_{t\in\mathbb{R}}$ with ACF $R_X(\tau)$

DEF The PSD is the Fourier transform of the ACF

$$S_X(\omega)=\int_{-\infty}^{+\infty} e^{-j\omega\tau} R_X(\tau)\,d\tau=(\mathcal{F} R_X)(\omega),\ \omega\in\mathbb{R}.$$

Fact By the inversion formula it follows that

$$R_X(\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\omega\tau} S_X(\omega)\,d\omega=(\mathcal{F}^{-1} S_X)(\tau).$$

Example Recall that $X(t)=A\sin(\omega_0 t+\Theta)$ with $\omega_0\in\mathbb{R}$ a constant and $A$ and $\Theta$ independent random variables with $\Theta$ unif$[0,2\pi]$ has $R_X(\tau)=\frac{1}{2}E(A^2)\cos(\omega_0\tau)$ which gives $S_X(\omega)=\frac{\pi}{2}E(A^2)\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right)$ by "the" backway method".

Example One common ACF (it turns out) is $R_X(\tau)=e^{-\alpha|\tau|}$ has $S_X(\omega)=\frac{2\alpha}{\alpha^2+\omega^2}$ since

$$\int_{-\infty}^{+\infty} e^{-j\omega\tau}e^{-\alpha|\tau|}\,d\tau=\int_0^{+\infty} e^{-(\alpha+j\omega)\tau}\,d\tau+\int_{-\infty}^{0} e^{(\alpha-j\omega)\tau}\,d\tau$$

$$=\left[-\frac{e^{-(\alpha+j\omega)\tau}}{\alpha+j\omega}\right]_{\tau=0}^{\tau=+\infty}+\left[\frac{e^{(\alpha-j\omega)\tau}}{\alpha-j\omega}\right]_{\tau=-\infty}^{\tau=0}=\frac{1}{\alpha+j\omega}+\frac{1}{\alpha-j\omega}$$

From this in turn one can conclude without calculation that $R_X(\tau)=\frac{2\alpha}{\alpha^2+\tau^2}$ has $S_X(\omega)=2\pi\, e^{-\alpha|\omega|}$.

For a discrete time WSS process $\{X(n)\}_{n\in\mathbb{Z}}$ with ACF $R_X(\tau)$ the PSD is given by

$$S_X(\omega)=(\mathcal{F}R_X)(\omega)=\sum_{k=-\infty}^{+\infty} e^{-j\omega k}R_X(k) \quad \text{for } \omega\in[-\pi,\pi].$$

By Fourier series technique it follows that

$$R_X(\tau)=(\mathcal{F}^{-1}S_X)(\tau)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{j\omega\tau}S_X(\omega)\,d\omega,\ \tau\in\mathbb{Z}$$

$\boxed{\text{Properties}}$ $S_X(\omega)\geq 0$

$S_X(\omega)=S_X(-\omega)$

$S_X(\omega)$ is real

$$E(X(t)^2)=\begin{cases}\frac{1}{2\pi}\int_{-\infty}^{+\infty}S_X(\omega)\,d\omega & \text{continuous time}\\ \frac{1}{2\pi}\int_{-\pi}^{\pi}S_X(\omega)\,d\omega & \text{discrete time}\end{cases}$$

Proof The first property is very hard to prove while the fourth is by inspection of the inversion formula. In continuous time ~~we~~ further have

$$S_X(-\omega)=\int_{-\infty}^{+\infty}e^{j\omega\tau}R_X(\tau)\,d\tau=\int_{-\infty}^{+\infty}e^{-j\omega\tau}R_X(-\tau)\,d\tau=S_X(\omega)$$

$$\overline{S_X(\omega)}=S_X(-\omega)=S_X(\omega)\ \#$$

$\boxed{\text{DEF}}$ The crosspectraldensity $S_{XY}(\omega)$ between two jointly WSS processes $X(t)$ and $Y(t)$ is defined as $S_{XY}(\omega)=(\mathcal{F}R_{XY})(\omega)$ where $R_{XY}(\tau)=E(X(t)Y(t+\tau))$.

The bandwith of a process is a certain measure of the width of the graph of $S_{XX}(f)$, for example, the width of the 3dB-zone. There are several ways to measure bandwidth.

Recall WSS $X(t)$ and $Y(t)$ are jointly WSS if $R_{XY}(t,t+\tau)$ does not depend on $t$.

**Exa** Now let $\{e[n]\}_{n=-\infty}^{+\infty}$ be IID or uncorrelated zero-mean with variance $\sigma^2$ ($\equiv$ discret white noise) and consider a process $\{Y[n]\}_{n=-\infty}^{+\infty}$ given by

$$Y[n] = \sum_{i=1}^{p} a_i Y[n-i] + \sum_{i=0}^{q} b_i e[n-i]$$

where $e[n]$ independent of earlier $Y[n]$

This process is called an ARMA(p,q)-process. When $p=0$ it is called an MA(q)-process and when $q=0$ it is called an AR(p)-process.

Let us study the AR(1)-process

$$Y[n] = a_1 Y[n-1] + e_n$$

We will see in Sec 6.5 that $Y$ is WSS so

$$R_Y(0) = E(Y[n]^2) = E((a_1 Y[n-1]+e_n)^2) = a_1^2 R_Y(0) + \sigma^2$$

so that $R_Y(0) = \frac{\sigma^2}{1-a_1^2}$. Further

$$R_Y(k) = E(Y(n)Y(n+k))$$

for $k \geq 1$

$$= E\Big(Y[n]\big(a_1 Y(n+k-1) + e_{n+k}\big)\Big) = a_1 R_Y(k-1)$$

which is a difference equation with solution

$$R_Y(k) = a_1^{|k|} R_Y(0) = a_1^{|k|} \frac{\sigma^2}{1-a_1^2}.$$

**DEF** White noise is a (zero-mean) WSS (possibly Gaussian) process $X(t)$ with constant PSD $S_X(\omega) = \sigma^2$.

Going the "backway" we realize that $X(t)$ must have ACF $R_X(\tau) = \sigma^2 \delta(\tau)$ in both discrete and continuous time.

# Section 6.5: LTI systems

**DEF** A linear time invariant (LTI) system with insignal $x(t)$ and outsignal $y(t) = (Tx)(t)$ satisfies

* $(T(\alpha x_1 + \beta x_2))(t) = \alpha (Tx_1)(t) + \beta (Tx_2)(t)$
* $(T(x(\cdot - t_0)))(t) = (Tx)(t - t_0)$

**DEF** The <u>impulse response</u> of an LTI system is $h(t) = (T\delta)(t)$.

**THM** For an LTI system we have

$$(Tx)(t) = (h * x)(t) = \begin{cases} \int_{-\infty}^{+\infty} h(t-u)x(u)\,du & \text{continuous time} \\ \sum_{k=-\infty}^{+\infty} h(t-k)\,x(k) & \text{discrete time} \end{cases}$$

<u>Proof</u> (Discrete time) As $x(t) = \sum_{k=-\infty}^{+\infty} x(k)\,\delta(t-k)$ we have

$$(Tx)(t) = T\left(\sum_{k=-\infty}^{+\infty} x(k)\,\delta(t-k)\right) = \sum_{k=-\infty}^{+\infty} x(k)\,T(\delta(t-k)) = (h * x)(t) \quad \#$$

We will now use a WSS process $X(t)$ with mean $\mu_X$ and ACF $R_X(\tau)$ as insignal to an LTI system. For the outsignal $Y(t)$ we then obtain

$$\mu_Y(t) = E(Y(t)) = E\left(\int_{-\infty}^{+\infty} X(u)\,h(t-u)\right) = \int_{-\infty}^{+\infty} E(X(u))\,h(t-u)\,du = \mu_X \int_{-\infty}^{+\infty} h(u)\,du$$

$$R_Y(t, t+\tau) = E(Y(t), Y(t+\tau)) = E\left(\left(\int_{-\infty}^{+\infty} h(u)\,X(t-u)\,du\right)\left(\int_{-\infty}^{+\infty} h(v)\,X(t+\tau-v)\,dv\right)\right)$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(u)\,h(v)\,R_X(\tau - v + u)\,du\,dv = (h(-\cdot) * h * R_X)(\tau) = R_Y(\tau)$$

$$R_{XY}(t,t+\tau) = E(X(t)Y(t+\tau)) = E\left(X(t)\int_{-\infty}^{+\infty} h(u)X(t+\tau-u)du\right)$$

$$= \int_{-\infty}^{+\infty} h(u)R_X(\tau-u)\,du = (h * R_X)(\tau)$$

$\boxed{\text{Def}}$ Frequency response (= transfer function)

$H(\omega) = (\mathcal{F}h)(\omega)$ in both continuous and discrete time

It follows that $S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$,

$S_{XY}(\omega) = H(\omega)S_X(\omega)$ and $S_{YX}(\omega) = \overline{H(\omega)}S_X(\omega)$

Here we made use of the very important fact that

$(\mathcal{F}(f*g)) = (\mathcal{F}f)(\mathcal{F}g)$ and $\mathcal{F}(f(-\cdot)) = \overline{\mathcal{F}f}$

Ex: (cont) An AR(1)-process $Y(k) = aY(k-1) + e(k)$ where $(e_k)_{k=-\infty}^{+\infty}$ is white noise can be considered as an LTI system with insignal $(e_k)_{k=-\infty}^{+\infty}$ and outsignal $Y(k)$ where

$$H(\omega)e(\omega) = (\mathcal{F}(h*e))(\omega) = Y(\omega) = a\sum_{k=-\infty}^{+\infty} e^{-j\omega k} Y(k-1) + e(\omega)$$

$$= a e^{-j\omega} Y(\omega) + e(\omega) = a e^{-j\omega} H(\omega)e(\omega) + e(\omega)$$

giving $H(\omega) = 1/(1 - a e^{-j\omega})$ so that

$S_Y(\omega) = |H(\omega)|^2 S_e(\omega) = \sigma^2/(1 + a^2 - 2a\cos(\omega))$.